MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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Dr. A. C. D. CROMMELIN, PRESIDENT, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Rev. Dr. Paul A. McNally, S.J., Director of the Georgetown College Observatory, Washington, D.C., U.S.A. ; and

Cyril Young, c/o Sir Howard Grubb, Parsons & Co., Heaton Works, Newcastle-on-Tyne.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

William Hunter McCrea, B.A., Trinity College, Cambridge (proposed by E. A. Milne) ;

Edgar Girard Croker Poole, M.A., D.Phil., Fellow and Tutor of New College, Oxford (proposed by H. H. Turner) ;

William Arthur Doubleday Smith, B.Sc., 35 Griffin Road, London, S.E. 18 (proposed by L. J. Comrie) ;

Arthur Thomas Wadley, M.Sc., 11 Tyrwhitt Road, London, S.E.4 (proposed by L. J. Comrie) ; and

The Rev. Eber Edward Wright, 115 Carlton Hill, Nottingham (proposed by W. M. Hunt).

One hundred and thirty-nine presents were announced as having been received since the last ordinary meeting, including, amongst others :—

Albany, Dudley Observatory : San Luis Catalogue of 15,333 stars for the epoch 1910.

Astrographic Catalogue : Bordeaux Observatory, zones +12°, +14° (9 charts) ; San Fernando Observatory, zone −3° (10 charts) ; Toulouse Observatory, zones +5°, +7°, +9° (13 charts).

A. E. Conrady : Applied Optics and Optical Design.

Strasbourg Observatory : Annales, Tome ii, fasc. 1.

A 2-inch telescope by Tully, formerly belonging to Francis Baily (presented by Lt.-Col. John Day, R.E.).

A Method adapted to Calculating Machines for finding the Paths of Comets and Minor Planets from Short Arcs. By R. T. A. Innes, D.Sc.

The formulæ at present in common use for finding orbits are not convenient for application with the electric or even hand-driven calculating machine. This point has been dealt with in Merton's paper in the *M.N.*, 1925 June, to which many references will be made, and Merton refers also to an earlier paper on the same subject by Veithen.

In the method now advocated, no assumption as to the form of the path is made, but as usual, it is assumed that the Sun is the only attracting body. It is however easy to include, if necessary, the accelerations due to the planets, as indeed is done by the Leningrad school. Equatorial rectangular co-ordinates are used from the earliest moment. Logarithmic and trigonometrical tables, collections of formulæ, etc., are not necessary, although a table such as Barlow's Squares and Reciprocals is useful, and occasionally a table of logarithms is convenient.

Frequent reference has been made to the following works : Moulton, *Celestial Mechanics*, 2nd edition, 1914, especially to article 141 ; Andoyer, *Mécanique Céleste*, 1923, i., article 25 ; Bauschinger, *Bahnbestimmung*, 1st edition, 1906 ; Plummer, *Dynamical Astronomy*, 1918 ; Watson, *Theoretical Astronomy*, 1868.

As to the types and capabilities of calculating machines, see Comrie's papers, *M.N.*, 1925 June, etc.

As remarked by all the authorities, the accuracy attainable in a first orbit is small. A few examples may be quoted :—

	a.	*e.*	
1036 Ganymede :			
Merton, 1924 Oct. 23–34 . .	2·523	0·5154	*M.N.*
Stracke, 1924, 7 normal places .	2·668	0·5394	*A.N.*, 5410.
79 Eurynome :			
Watson, 1863 Sept. 15–28 . .	2·426	0·1884	*Th. Astr.*
,, 1863–4, 4 normal places	2·444	0·1953	,,
841 Arabella :			
Berberich, 1916 Oct. 1–Nov. 26 .	2·250	0·0646	*L.O.B.*, 406.
Makemson, 1928 Feb. 24–Apl. 22	2·255	0·0694	,,

It is evident that even the second significant figure is often doubtful.

The fact that no assumption as to the form of the orbit is made calls for some comment. The usual solution assumes that the body moves in either an ellipse or a parabola with the Sun in one focus and that it describes equal areas in equal times. All this is unnecessary as it is already implicit in the Newtonian law.

The method to be detailed here has now been applied to four cases, one being a minor planet, the other three being comets. By a strange chance, not one of these three comets moves in a parabola. The first calculation for Comet 1927*f* yielded an elliptical orbit, and a lucky guess indicated a period of 11·1 years (see *A.N.*, 5525). Dr. Crommelin's later and more complete calculations confirm the period closely enough (see *M.N.*, 1927 November, p. 87).

The first orbit for Comet 1927*b* (*A.N.*, 5530) indicated a hyperbolic orbit, and this was also confirmed later by Dr. Crommelin (see *J.B.A.A.*, 1928 March, p. 187). The case of Comet 1925*c* is one of the numerical examples of this paper and a hyperbolic orbit is indicated by the first trial. This has also been confirmed later as Goryatschew's definitive orbit is a hyperbola (see *M.N.*, 1928 June, p. 707). The method has so far been successful in practice.

If a comet did actually move in a parabola, then undoubtedly a valuable equation of condition is introduced, and it will serve to smooth over the errors of observation. It is doubtful if comets do move in parabolas. In finding a parabolic orbit it is always assumed that the Sun is the only attracting force, whereas the comet approaches the Sun under the action of the whole solar system, so that the accelerations should be multiplied by

$$\Sigma(1 + m_i) \qquad i = 1 \text{ to } 8 \text{ for Mercury to Neptune.}$$

If the motion is elliptical, then the neglect of the other attractions make the motion seem greater and an elongated ellipse may seem to be a parabola by this neglect.

The method is self-contained and can be written out *de novo* in a few minutes. No trigonometrical functions are used, so that the equations are simple and straightforward. A fair number of check-equations is available.

It is well to bear in mind exactly what is required in seeking a solution, namely the heliocentric co-ordinates and their velocities at a given moment, that is to say

$$x, y, z \quad \text{and} \quad \dot{x}, \dot{y}, \text{and } \dot{z}.$$

Once these are obtained, the gravitational path of the object at all times, past, present, or future, can be found.

From these co-ordinates and velocities, orbit elements are easily derived, and with a wide choice of methods. One such is given in Appendix i. to Merton's paper on Grigg's Comet (*Mems.*, *R.A.S.*, **64**, 103), but it makes considerable use of trigonometrical functions. The method used in the present paper uses none and is therefore better for work with the calculating machine.

Three convenient dates, on which reliable observations have been made, are to be chosen for the initial solution. In short, the preliminary steps do not vary from the usual modern practice.

From the observations corrected as indicated in the text-books (or see Merton, i. 717) compute for each date, t_1, t_2, and t_3.

$$\begin{aligned} l &= \mathrm{C}\delta\mathrm{C}a, \\ m &= \mathrm{C}\delta\mathrm{S}a, \\ n &= \mathrm{S}\delta, \end{aligned}$$

and the solar co-ordinates X, Y, and Z (the *B.J.* is the most convenient source) referred to the same equinox and corrected for parallax (see Merton, p. 695). The epoch to be chosen as origin is the date t_2 of the middle observation. The suffixes $_1$, $_2$, and $_3$ will indicate the dates.

In the examples given, far more figures are shown than need be written in actual practice. Thus, for such an expression as squaring the co-ordinates x, y, and z to find r^2, the result r^2 is totalled on the machine, and x^2, etc., need not be shown. A rough count of the number of figures involved gives 2400, whilst a count for a very similar calculation by the Gaussian method (Bauschinger's example, pp. 282–285) gives 2700, to say nothing of the involved formulæ and the almost constant reference to tables.

The heliocentric movement near the middle date will then be

$$\begin{aligned} x_1 &= x - \dot{x}\tau_3 + \ddot{x}\frac{\tau_3^2}{2} - \dddot{x}\frac{\tau_3^3}{\lfloor 3} + x^{\mathrm{iv}}\frac{\tau_3^4}{\lfloor 4} -, \text{ etc.} \\ x_3 &= x + \dot{x}\tau_1 + \ddot{x}\frac{\tau_1^2}{2} +, \text{ etc.,} \end{aligned}$$

with similar series for the y and z co-ordinates.

In these

$$\ddot{x} = -kx/r^3$$

and by differentiation

$$\dddot{x} = -k\dot{x}/r^3 + 3kx\dot{r}/r^4,$$

and this can be carried on indefinitely, but whenever $\ddot{x}$ occurs in further differentiations, it will be replaced by $-kx/r^3$ (Watson, p. 174).

If we put

$$\begin{aligned} f_1 &= 1 - \frac{k}{2}\frac{\tau_3^2}{r^3} - \frac{k}{2}\frac{\dot{r}}{r^4}\tau_3^3 +, \text{ etc.} \\ g_1 &= \tau_3\left(1 - \frac{k}{6}\frac{\tau_3^2}{r^3} - \frac{k}{4}\frac{\dot{r}}{r^4}\tau_3^3 + \ldots\right) \\ f_2 &= 1 - \frac{k}{2}\frac{\tau_1^2}{r^3} + \frac{k}{2}\frac{\dot{r}}{r^4}\tau_1^3 +, \text{ etc.} \\ g_2 &= \tau_1\left(1 - \frac{k}{6}\frac{\tau_1^2}{r^3} + \frac{k}{4}\frac{\dot{r}}{r^4}\tau_1^3 + \ldots\right) \end{aligned}$$

we can write

$$\begin{aligned} x_1 &= xf_1 - \dot{x}g_1 \\ x_3 &= xf_3 + \dot{x}g_3. \end{aligned}$$

All this is commonplace and will be found in all the treatises; see, for example, Bauschinger, p. 175, or for closed expressions for f and g, p. 173.*

We have then

$$\begin{array}{lllll} t_1 & x_1 = xf_1 - \dot{x}g_1 & = \rho_1 l_1 - \mathrm{X}_1 \\ t_2 & x_2 = x & = \rho_2 l_2 - \mathrm{X}_2 \\ t_3 & x_3 = xf_3 + \dot{x}g_3 & = \rho_3 l_3 - \mathrm{X}_3. \end{array}$$

Multiply these equations in order by g_3, $-(f_1 g_3 + f_3 g_1)$ and g_1, and to abbreviate replace $f_1 g_3 + f_3 g_1$ by g_2, then add, and do similarly with the equations in y and z.

This done, we have

$$\left.\begin{array}{l} \rho_1 l_1 g_3 - \rho_2 l_2 g_2 + \rho_3 l_3 g_1 = g_3 \mathrm{X}_1 - g_2 \mathrm{X}_2 + g_1 \mathrm{X}_3 \\ \rho_1 m_1 g_3 - \rho_2 m_2 g_2 + \rho_3 m_3 g_1 = g_3 \mathrm{Y}_1 - g_2 \mathrm{Y}_2 + g_1 \mathrm{Y}_3 \\ \rho_1 n_1 g_3 - \rho_2 n_2 g_2 + \rho_3 n_3 g_1 = g_3 \mathrm{Z}_1 - g_2 \mathrm{Z}_2 + g_1 \mathrm{Z}_3 \end{array}\right\} \text{See Merton, (9), on p. 695.}$$

Divide by g_2 and put $\frac{g_3}{g_2} = c_1$ and $\frac{g_1}{g_2} = c_3$.

Put

$$\mathrm{A} = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \qquad \mathrm{B}_i = \begin{vmatrix} \mathrm{X}_i & l_2 & l_3 \\ \mathrm{Y}_i & m_2 & m_3 \\ \mathrm{Z}_i & n_2 & n_3 \end{vmatrix}$$

$$\mathrm{C}_i = \begin{vmatrix} l_1 & \mathrm{X}_i & l_3 \\ m_1 & \mathrm{Y}_i & m_3 \\ n_1 & \mathrm{Z}_i & n_3 \end{vmatrix} \qquad \mathrm{D}_i = \begin{vmatrix} l_1 & l_2 & \mathrm{X}_i \\ m_1 & m_2 & \mathrm{Y}_i \\ n_1 & n_2 & \mathrm{Z}_i \end{vmatrix} \qquad i = 1, 2, 3,$$

then the solution of the equations is

$$\left.\begin{array}{rl} \rho_1 \mathrm{A} c_1 = & c_1 \mathrm{B}_1 - \mathrm{B}_2 + c_3 \mathrm{B}_3 \\ \rho_2 \mathrm{A} = & -c_1 \mathrm{C}_1 + \mathrm{C}_2 - c_3 \mathrm{C}_3 \\ \rho_3 \mathrm{A} c_3 = & c_1 \mathrm{D}_1 - \mathrm{D}_2 + c_3 \mathrm{D}_3 \end{array}\right\} \text{See Bauschinger, p. 259 (261 in 2nd edition), or Moulton, p. 250.}$$

The 10 determinants A, B_i, C_i, and D_i are independent of the system of axes which may be adopted.

The values of c_1 and c_3 can be derived from the f and g series already given, but we carry their values from the text-books. Very exactly we have the well-known formulæ of Gibbs: †

$$\left.\begin{array}{ll} c_1 = c_1{}^0\left(\frac{1 + \mu_1 r_1{}^{-3}}{1 - \mu_2 r_2{}^{-3}}\right) & c_1{}^0 = \frac{\tau_1}{\tau_2} \\ c_3 = c_3{}^0\left(\frac{1 + \mu_3 r_3{}^{-3}}{1 - \mu_2 r_2{}^{-3}}\right) & c_3{}^0 = \frac{\tau_3}{\tau_2} \end{array}\right\} \text{See Plummer, p. 63, where they are neatly derived on gravitational principles only.}$$

* In second edition see pp. 177 and 175 respectively.

† Merton, p. 699, gives the references.

in which

$$\mu_1 = \frac{k}{12}(\tau_2\tau_3 - \tau_1^2)$$

$$\mu_2 = \frac{k}{12}(\tau_1\tau_3 + \tau_2^2)$$

$$\mu_3 = \frac{k}{12}(\tau_1\tau_2 - \tau_3^2).$$

In the first applications of c_1 and c_3 we have to confuse r_1 and r_3 with r_2, and we have then as an approximation

$$c_1 = c_1^0 + \nu_1/r_2^3 \qquad \nu_1 = c_1^0\frac{(\tau_2^2 - \tau_1^2)}{6}$$

$$c_3 = c_3^0 + \nu_3/r_2^3 \qquad \nu_3 = c_3^0\frac{(\tau_2^2 - \tau_3^2)}{6}.$$

Once ρ_2 is known,

$$\rho_1 = -\frac{\rho_2}{c_1}\frac{(c_1B_1 - B_2 + c_3B_3)}{(c_1C_1 - C_2 + c_3C_3)},$$

and similarly for ρ_3.

It may be objected that the determinants look formidable, but in reality they are very easy to compute, especially so as no tables have to be referred to, and they contain nothing but the mere data of observation.

To find ρ_2 we rewrite the equation for it, in the form

$$\rho_2 A = -c_1^0C_1 + C_2 - c_3^0C_3 - \frac{k}{r_2^3}(\nu_1C_1 + \nu_3C_3).$$

We also have

$$r_2^2 = \rho_2^2 - 2\rho_2R_2C\psi + R_2^2.$$

From these two equations ρ_2 and r_2 must be found by the usual method of trials.

With ρ_2 we compute ρ_1 and ρ_3, and then the time-corrections and first approximations to the values of r_1 and r_3. We can then compute c_1 and c_3 more exactly and find better values of ρ_1, ρ_2, and ρ_3. The additional accuracy so gained is generally illusory, for as Merton says, "*In most cases, therefore, the first solution of* ρ_2 *is final.*" It is now possible to bring the values of ρ_1 and ρ_3 into an exact (or at least nearly exact) dynamical agreement, as will be shown in both examples, by computing the heliocentric co-ordinates x_i, y_i, and z_i, and adjusting them by varying ρ_1 and ρ_3 until equations of the type

$$c_1x_1 + c_3x_3 = x_2$$

are exactly satisfied. If one or more of the observations is in error, it may be impossible to effect a complete reconciliation. Having done this, it remains to find the velocities $\dot{x}$, $\dot{y}$, and $\dot{z}$. Each of the equations

$$x_1 = xf_1 - \dot{x}g_1$$
$$x_3 = xf_3 + \dot{x}g_3, \text{ etc.},$$

will furnish a value of $\dot{x}$, $\dot{y}$, and $\dot{z}$, and their agreement, or lack of it, will be an indication.

The series to be used for the f_1 and g_1 are

$$f_1 = 1 - \frac{k}{2}\tau_3{}^2\left[\frac{1}{r_2{}^3} + \frac{c_3{}^0}{3}\left(\frac{1}{r_1{}^3} - \frac{1}{r_3{}^3}\right)\right]$$

$$g_1 = \tau_3\left\{1 - \frac{k}{6}\tau_3{}^2\left[\frac{1}{r_2{}^3} + \frac{c_3{}^0}{2}\left(\frac{1}{r_1{}^3} - \frac{1}{r_3{}^3}\right)\right]\right\},$$

and for f_3 and g_3 replace τ_3 by τ_1 and $c_3{}^0$ by $-c_1{}^0$. As a check, compute

$$g_2 = f_1 g_3 + f_3 g_1,$$

and then

$$c_1 = \frac{g_3}{g_2} \qquad c_3 = \frac{g_1}{g_2},$$

which should agree with the values previously found.

A Cowell ephemeris can now be started, but the possibility that the object may be a periodic comet or a known planet justifies an attempt at a conic-section orbit. If this should turn out to be hyperbolic, or parabolic, with a the semi-axis major very large, it will be useless to go further because it is much simpler to compute a Cowell ephemeris than to have to refer to the tables for hyperbolic or parabolic motion. If the motion is elliptical, then the final form will be

$$\mathrm{M} = \mathrm{M}_0 + \mu^0(t - \text{epoch}) \quad \text{and} \quad e.$$

With e and M, X and Y can be interpolated for any excentricity from the Union Observatory Tables (Appendix to *U.O.C.* No. 71), or if the excentricity is less than 0·423 (Sin 25°) then Stracke's Tables (Berlin, *Astr. Rechen-Inst.*, No. 46) give X and Y (in his notation C and S) to 4 places of decimals.

Then

$$x = a\mathrm{P}_x\mathrm{X} + a\mathrm{Q}_x\mathrm{Y}$$
$$y = a\mathrm{P}_y\mathrm{X} + a\mathrm{Q}_y\mathrm{Y}$$
$$z = a\mathrm{P}_z\mathrm{X} + a\mathrm{Q}_z\mathrm{Y}.$$

It will be remarked that throughout no trigonometrical functions have been used, not because their use has been avoided, but simply because they do not appear naturally.

The formulæ for deriving the orbit-elements and the P's and Q's are shown in the two examples which, with Dr. Merton's permission, are built up on the observational data used in his paper.

It is hardly likely that the formulæ used cannot be improved upon, but it may be claimed that they are straightforward and very easily computed with the machine. The whole of the formulæ can be written on a small card. In the examples to be given, far too many decimals have been employed, but this is a non-costly luxury with a calculating machine.

Nothing is said about a double solution or about how to correct the first solution by using more observations. A double solution appears

in one of the cases quoted (Comet 1927*f*) (see Crommelin, *M.N.*, 1927 November, p. 85, etc.) and may be dealt with later.

The examples are so detailed that no references to formulæ are required. In fact every figure used is shown so that the reader can easily follow the whole process.

Example (1).—1036 *Ganymede.*

I. *The data given by observations.*

The R.A. and Dec. will generally be given for the mean equinox of the beginning of the year and may require correction for stellar aberration.

	1924 G.M.A.T.	R.A.	Dec.	θ.
		h m s	° ′ ″	h m
t_1	October 23·30940	21 5 11·60	+ 15 28 33·8	22 14·2
t_2	,, 28·47775	21 30 27·23	+ 12 4 32·5	1 56·1
t_3	,, 34·31383	21 58 32·11	+ 8 24 29·4	22 23·0

	d		d
$\tau_3 = t_2 - t_1 =$	5·16835	$\tau_3^2 =$	26·709
$\tau_1 = t_3 - t_2 =$	5·83608	$\tau_1^2 =$	34·059
$\tau_2 = t_3 - t_1 =$	11·00443	$\tau_2^2 =$	121·097

$$c_1^0 = \frac{\tau_1}{\tau_2} = 0{\cdot}530339$$

$$c_3^0 = \frac{\tau_3}{\tau_2} = 0{\cdot}469661$$

$$\nu_1^0 = c_1^0 \times \frac{k}{6}(\tau_2^2 - \tau_1^2) = {\cdot}002277$$

$$\nu_3^0 = c_3^0 \times \frac{k}{6}(\tau_2^3 - \tau_3^3) = {\cdot}002186.$$

$$\frac{k}{6} = {\cdot}0000493187.$$

k = Square of the Gaussian constant.

II. *Solar Co-ordinates from the ephemeris, with corrections for parallax.*

With the local sidereal time θ (see I.) compute the effect of parallax ΔX, ΔY, and ΔZ, using Merton's (7), p. 695, and his table, p. 728.*

	ΔX.	ΔY.	ΔZ.	X.	Y.	Z.
(1)	− 228	+ 113	− 341	− ·860286	− ·457649	− ·198547
(2)	− 233	− 129	− 332	− ·810811	− ·526063	− ·228209
(3)	− 243	+ 109	− 332	− ·747072	− ·598100	− ·259468

III. *Planet's Direction-Cosines and the Triangle Sun-Earth-Planet.*

$l = \mathrm{C}\delta\mathrm{C}\alpha$ $\qquad \mathrm{RC}\psi = l\mathrm{X} + m\mathrm{Y} + n\mathrm{Z}$

$m = \mathrm{C}\delta\mathrm{S}\alpha$

$n = \mathrm{S}\delta$ $\qquad \mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2.$

* Note *erratum*: for Arequipa $\Delta Z = +120$.

(1)	(2)	(3)
l + ·696735	+ ·776976	+ ·853538
m − ·665853	− ·593753	− ·500091
n + ·266836	+ ·209204	+ ·146224
2RCψ − ·695288	− ·730743	− ·752981
R^2 + ·988956	+ ·986236	+ ·983164.

N.B.—In I., II., and III. the figures are taken without change from Merton, p. 710.

With the computation of l, m, and n, trigonometrical tables are done with until a R.A. and Dec. ephemeris is made.

IV. *The* 10 *Determinants.*

$$A = l_2(m_3n_1 - m_1n_3) + m_2(l_1n_3 - l_3n_1) + n_2(l_3m_1 - l_1m_3)$$

$$B_i = X_i(m_2n_3 - m_3n_2) + Y_i(l_3n_2 - l_2n_3) + Z_i(l_2m_3 - l_3m_2)$$
$$C_i = X_i(m_3n_1 - m_1n_3) + Y_i(l_1n_3 - l_3n_1) + Z_i(l_3m_1 - l_1m_3)$$
$$D_i = X_i(m_1n_2 - m_2n_1) + Y_i(l_2n_1 - l_1n_2) + Z_i(l_1m_2 - l_2m_1)$$

$$i = 1, 2, 3.$$

$m_2n_3 - m_3n_2$ + ·0178001	$l_3n_2 - l_2n_3$ + ·0649510	$l_2m_3 - l_3m_2$ + ·1182321
$m_3n_1 - m_1n_3$ − ·0360783	$l_1n_3 - l_3n_1$ − ·1258753	$l_3m_1 - l_1m_3$ − ·2198999
$m_1n_2 - m_2n_1$ + ·0342192	$l_2n_1 - l_1n_2$ + ·0615655	$l_1m_2 - l_2m_1$ + ·1036635

− ·02803220
+ ·07473888
− ·04600398

A + ·0007027 1/A + 1423·1.

(1)	(2)	(3)
− ·0153132	− ·0144325	− ·0132979
− ·0297248	− ·0341683	− ·0388472
− ·0234746	− ·0269817	− ·0306775
B_1 − ·0685126	B_2 − ·0755825	B_3 − ·0828226
+ ·0310379	+ ·0292529	+ ·0269533
+ ·0576067	+ ·0662184	+ ·0752861
+ ·0436605	+ ·0501832	+ ·0570570
C_1 + ·1323051	C_2 + ·1456545	C_3 + ·1592964
− ·0164621	− ·0155154	− ·0142957
− ·0281754	− ·0323873	− ·0368223
− ·0205820	− ·0236569	− ·0268973
D_1 − ·0652195	D_2 − ·0715596	D_3 − ·0780153

These determinants remain unchanged throughout the whole calculation. Their values should be carefully checked. The value of A can be computed directly from the given spherical co-ordinates

$$A = C\delta_1 C\delta_2 C\delta_3[T\delta_1 S(\alpha_3 - \alpha_2) + T\delta_2 S(\alpha_1 - \alpha_3) + T\delta_3 S(\alpha_2 - \alpha_1)].$$

As usual, **S**, **C**, and **T** stand for sin, cos, and tan.

V. *Equations for* ρ_2 *and* r_2.

$$C_i' = C_i/A.$$

$C_1' + 188{\cdot}281$	$\times -C_1^0$	$= -\ 99{\cdot}8528$	$\times -\nu_1^0$	$= -\ 0{\cdot}42863$
$C_2' + 207{\cdot}2783$		$= +\ 207{\cdot}2783$		
$C_3' + 226{\cdot}692$	$\times -C_3^0$	$= -\ 106{\cdot}4683$	$\times -\nu_3^0$	$= -\ 0{\cdot}49560$
		$+\ 0{\cdot}9572$		$-\ 0{\cdot}92423$

$$\log\ 9{\cdot}96578n.$$

So
$$\rho_2 = 0{\cdot}9572 - 0{\cdot}92423/r_2^3$$
$$r_2^2 = \rho_2^2 - 0{\cdot}730743\rho_2 + 0{\cdot}986236.$$

Trials.

ρ_2	$0{\cdot}5$	$0{\cdot}502$
ρ_2^2	$0{\cdot}25$	$0{\cdot}25200$
$2\rho_2 RC\psi$	$0{\cdot}36537$	$0{\cdot}36683$
R_2^2	$0{\cdot}98624$	$0{\cdot}98624$
r_2^2	$1{\cdot}60161$	$1{\cdot}60507$
$\log r_2^2$	$0{\cdot}20456$	$0{\cdot}20550$
$\frac{1}{2}$	$0{\cdot}10228$	$0{\cdot}10275$
	$0{\cdot}30684$	$0{\cdot}30825$
	$9{\cdot}96578$	$9{\cdot}96578$
	$9{\cdot}65894$	$9{\cdot}65738$
	$0{\cdot}45598$	$0{\cdot}45451$
	$0{\cdot}95720$	$0{\cdot}95720$
ρ_2	$0{\cdot}50122$	$0{\cdot}50269$

For $\rho_2 = 0{\cdot}50000$ excess is 122

$$-53$$

" $= 0{\cdot}50200$ " " 69.

So excess will vanish for $\dfrac{2 \times 69}{53}$

or for

$$\rho_2 = 0{\cdot}5020 + 0{\cdot}0026.$$

Then

ρ_2	$0{\cdot}5046$
	$0{\cdot}25462$
	$0{\cdot}36873$
	$0{\cdot}98624$
r_2^2	$1{\cdot}60959$

log r_2^2	0·20672
,, r_2^2	0·10336
,, r_2^3	0·31008 . . . $1/r_2^3$ = 0·48969
	9·96578
	9·65570
	0·45259
	0·95720

ρ_2 = 0·50461.

N.B.—The important A is the difference of large numbers (six significant figures) and the first two vanish, whilst the last two are probably inaccurate because of errors of observation. It is therefore unlikely that the value of ρ_2 just found can be really improved upon. See Merton's remarks, p. 698.

VI. *Derivation of* ρ_1 *and* ρ_3.

Form c_1 and c_3.

	c_1^0 =	·530339		c_3^0 =	·469661
ν_1^0/r_2^3	=	1115	ν_3^0/r_2^3	=	1070
c_1		·531454	c_3		·470731

Then

c_1B_1 − ·0364113	− c_1C_1 − ·0703141	c_1D_1 − ·0346612
− B_2 + ·0755825	C_2 + ·1456545	− D_2 + ·0715596
c_3B_3 − ·0389872	− c_3C_3 − ·0749858	c_3D_3 − ·0367242
ΣB + ·0001840	ΣC + ·0003546	ΣD + ·0001742

$1/\Sigma C$ + 2820·1.

$\Sigma C/A = \rho_2$ = 0·50462. This is a verification.

Then	$\Sigma B/\Sigma C$ = ·51889	$\Sigma D/\Sigma C$ = ·49126
	$\rho_2 c_1$ = ·94951	ρ_2/c_3 = 1·07199.
Giving	ρ_1 = 0·49269	ρ_3 = 0·52663.

It will be seen that the sums of B, C, and D each also nearly cancel, leaving at the most two- or three-figure accuracy.

Then

	(1)	(2)	(3)
R^2	·98896	·98624	·98316
$2RC\psi\rho$	·34256	·36875	·39654
ρ^2	·24274	·25464	·27734
r^2	1·57426	1·60963	1·65704
r^3	1·9752	2·0422	2·1330
$1/r^3$	·50627	·48968	·46881.

VII. *Corrections for Light-time and corrected values of the time-intervals.*

	(1)	(2)	(3)
$0{\cdot}005770\times\rho$	− 283	− 291	− 303
t'	23·30657	28·47484	34·31080

$\tau_1 = 5{\cdot}83596$
$\tau_2 = 11{\cdot}00423$
$\tau_3 = 5{\cdot}16827$

$\tau_1^2 =$	34·05843		
$\tau_3^2 =$	26·71101		
$\tau_1\tau_2 - \tau_3^2 =$	37·50924	$\tau_1\tau_2 =$	64·22025
$\tau_2\tau_3 - \tau_1^2 =$	22·81440	$\tau_2\tau_3 =$	56·87283
$\tau_2^2 =$	121·09308	=	121·09308
$\tau_1\tau_3 =$	30·16182		
$\tau_1\tau_3 + \tau_2^2 =$	151·25490		

$c_1^0 = {\cdot}530338 \qquad c_3^0 = {\cdot}469662$

$$\frac{k}{12} = {\cdot}0000246593$$

$$\mu_1 = \frac{k}{12}(\tau_2\tau_3 - \tau_1^2)$$

$$\mu_2 = \frac{k}{12}(\tau_1\tau_3 + \tau_2^2)$$

$$\mu_3 = \frac{k}{12}(\tau_1\tau_2 - \tau_3^2).$$

	(1)		(2)	(3)
μ	·0005626		·0037298	·0009250
$\div r^3$	·0002848		·0018264	·0004336
		Square	33	
	·0018297	←—	·0018297 —→	·0018297
	1·0021145	Add unity to each and multiply by c_1^0 and c_3^0.		1·0022633
c_1	·531459			c_3 ·470725.

Merton's final values for c_1 and c_3 are ·531459 and ·470727 (p. 712, VIII.).

With these values the whole work in V. and VI. could be revised, but this refinement is not warranted. Actually it was done and the effect was to alter c_1 by 1 unit and c_3 by 2 units in the sixth decimal place.

We may, however, revise the work in VI. and this is done in VI*a*. which follows. It will be seen that the "improvement" is far less than the probable errors of the observations.

VI*a*.

c_1B_1 − ·0364116	− c_1C_1 − ·0703147	c_1D_1 − ·0346615
− B_2 + ·0755825	C_2 + ·1456545	− D_2 + ·0715596
c_3B_3 − ·0389867	− c_3C_3 − ·0749848	c_3D_3 − ·0367235
ΣB + ·0001842	ΣC + ·0003550	ΣD + ·0001743

$$\Sigma C/A = \rho_2 = 0{\cdot}50519.$$

$\Sigma B/\Sigma C$ = ·51887	$\Sigma D/\Sigma C$ = ·49099
ρ_2/c_1 = ·95057	ρ_2/c_3 = 1·07322
ρ_1 = ·49323	ρ_3 = ·52694.

Compare the sums and ratios of B, C, and D with those in VI. Then

	(1)	(2)	(3)
R^2	·98896	·98624	·98316
$2\rho RC\psi$	·34294	·36916	·39678
ρ^2	·24328	·25522	·27767
r^2	1·57518	1·61062	1·65761
$1/r^3$	0·50581	0·48923	0·46858

VIII. *Heliocentric Co-ordinates* $\left\{\begin{matrix} c_1 = \cdot 531459 \\ c_3 = \cdot 470725 \end{matrix}\right\}$.

ρ_1 = 0·49323	ρ_2 = 0·50519	ρ_3 = 0·52694	$c_1x_1 + c_3x_3$, etc.	
ρl + ·343651	+ ·392521	+ ·449763	·639843	
X − ·860286	− ·810811	− ·747072	·563380	
x + 1·203937	+ 1·203332	+ 1·196835	+ 1·203223	+ 109
ρm − ·328419	− ·299958	− ·263518	·068680	
Y − ·457649	− ·526063	− ·598100	·157496	
y + ·129230	+ ·226105	+ ·334582	+ ·226176	− 71
ρn + ·131616	+ ·105688	+ ·077051	·175468	
Z − ·198547	− ·228209	− ·259468	·158408	
z + ·330163	+ ·333897	+ ·336519	+ ·333876	+ 21

We can now bring ρ_1 and ρ_3 into correspondence with ρ_2.

$$\left.\begin{matrix} + \cdot 3437\frac{c_1}{\rho_1}\Delta\rho_1 + \cdot 4498\frac{c_3}{\rho_3}\Delta\rho_3 = + 108\ * \\ - \cdot 3284 \quad , \quad - \cdot 2635 \quad , \quad = - 71 \\ + \cdot 1316 \quad , \quad + \cdot 0771 \quad , \quad = + 21 \end{matrix}\right\} \text{Unit 6th decimal.}$$

* Error should be 109, but carried through as shown.

We have (VI*a*.) $\frac{c_1}{\rho_1} = 1{\cdot}0775$, $\frac{c_3}{\rho_3} = 0{\cdot}8933$, giving

$$\begin{aligned} +\,{\cdot}3703\Delta\rho_1 + {\cdot}4018\Delta\rho_3 &= +\,108 \\ -\,{\cdot}3539 \;,, \quad -\,{\cdot}2354 \;,, \quad &= -\,71 \\ +\,{\cdot}1418 \;,, \quad +\,{\cdot}0689 \;,, \quad &= +\,21. \end{aligned}$$

The first two give roughly

$$\begin{aligned} +\,{\cdot}1310\Delta\rho_1 + {\cdot}1418\Delta\rho_3 &= +\,38{\cdot}2 \\ -\,{\cdot}1310 \;,, \quad -\,{\cdot}0871 \;,, \quad &= -\,26{\cdot}2 \qquad \Delta\rho_3 = +\,0{\cdot}000220. \\ +\,{\cdot}0547 \;,, \quad &= +\,12{\cdot}0 \end{aligned}$$

Substituting

		$\Delta\rho_1$.
$+\,{\cdot}3703\Delta\rho_1$	$= +\,19{\cdot}6$	$+\,{\cdot}000053$
$-\,{\cdot}3539$,,	$= -\,19{\cdot}2$	54
$+\,{\cdot}1418$,,	$= +\,5{\cdot}8$	41
$+\,{\cdot}8660$	$+\,44{\cdot}6$	$+\,{\cdot}000051$ Mean,

and

		$\Delta\rho_3$.
$+\,{\cdot}4018\Delta\rho_3$	$= +\,89$	$+\,{\cdot}000221$
$-\,{\cdot}2354$,,	$= -\,53$	225
$+\,{\cdot}0689$,,	$= +\,14$	203
·7061	156	$+\,{\cdot}000221$ Mean.

So that for $\rho_2 = 0{\cdot}505190$, the gravitationally correct values of ρ_1 and ρ_3 are 0·493281 and 0·527161.

If these values of the ρ's are compared with Merton's (p. 712) it will be seen that they exceed them by

0·00069
and 0·00069
0·00070
respectively.

In other words, ρ_1 and ρ_3 are forced into agreement with ρ_2, but observations only furnish ρ_2 with two- or three-figure accuracy.

Correcting x_1, x_3, etc., for the new values of ρ_1 and ρ_3 we have

				$c_1\Delta x_1 + c_3\Delta x_3$.
$l_1\Delta\rho_1$	$= +\,{\cdot}000036$	$l_3\Delta\rho_3$	$= +\,{\cdot}000189$	$19 + 89 = +\,108$
m_1 ,,	$-\,29$	m_3 ,,	$-\,111$	$-\,15 - 52 = -\,67$
n_1 ,,	$+\,14$	n_3 ,,	$+\,32$	$7 + 15 = +\,22$

giving

	(1)	(2)	(3)	$c_1x_1 + c_3x_3$.
x	$+\,1{\cdot}203973$	$+\,1{\cdot}203332$	$+\,1{\cdot}197024$	1·203331
y	$+\,{\cdot}129201$	·226105	·334471	·226109
z	$+\,{\cdot}330177$	·333897	·336551	·333899.

The adjustment, although not perfect, more than suffices.

IX. *The Velocities $\dot{x}$, $\dot{y}$, and $\dot{z}$.*

Compute

$$f_1 = 1 - \frac{k}{2}\tau_3{}^2\left[\frac{1}{r_2{}^3} + \frac{c_3{}^0}{3}\left(\frac{1}{r_1{}^3} - \frac{1}{r_3{}^3}\right)\right] \qquad \frac{k}{2} = \cdot 0001147956.$$

$$g_1 = \tau_3\left\{1 - \frac{k}{6}\tau_3{}^2\left[\frac{1}{r_2{}^3} + \frac{c_3{}^0}{2}\left(\frac{1}{r_1{}^3} - \frac{1}{r_3{}^3}\right)\right]\right\}$$

For f_2 and g_2 replace τ_3 by τ_1 and $c_3{}^0$ by $-c_1{}^0$.

From VI*a*.

$$1/r_1{}^3 = 0{\cdot}50581$$
$$1/r_3{}^3 = 0{\cdot}46858$$
$$0{\cdot}03723$$

$\times c_3{}^0$	0·01975			$\times c_1{}^0$	0·01748			
$\div 3$	0·00658	$\div 2$	0·00988	$\div 3$	0·00583	$\div 2$	0·00874	
$1/r_2{}^3$	0·48923		0·48923		0·48923		0·48923	
	0·49581		0·49911		0·48340		0·48049	

$$\frac{k}{2}\tau_3{}^2 = \cdot 0039521 \qquad \frac{k}{6}\tau_3{}^2 = \cdot 0013174 \qquad \frac{k}{2}\tau_1{}^2 = \cdot 0050391 \qquad \frac{k}{6}\tau_1{}^2 = \cdot 0016797.$$

Multiplying

	·0019595	·0006575		·0024359	·0008071
$f_1 =$	0·9980405	·9993425	$f_3 =$	·9975641	·9991929
		$\times \tau_3$			$\times \tau_1$
		$g_1 = 5{\cdot}16487$			$g_3 = 5{\cdot}83125.$

Check

$$f_1 g_3 = 5{\cdot}81982$$
$$f_3 g_1 = 5{\cdot}15229$$
$$g_2 = 10{\cdot}97211, \quad \text{then} \quad \frac{g_3}{g_2} = c_1 = 0{\cdot}531461, \quad \frac{g_1}{g_2} = c_3 = 0{\cdot}470727$$

Compare VII., also Merton, VIII., p. 712.

x_1	1·203973	x_3	1·197024		
$x_2 f_1$	1·200974	$x_2 f_3$	1·200401		
$\dot{x} g_1$	2999	$\dot{x} g_3$	− 3377	yielding	− ·0005807
	·129201		334471		+ ·0186764
	225662		225554		·186781
	96461		108917	$\dot{y}$	+ ·0186772 ,,
	330177		336551		+ ·0005936
	333243		333084		5946
	3066		3467	$\dot{z}$	+ ·0005941 ,,

$$\dot{x} = -\cdot 0005799 \text{ Mean.}$$

X. *Orbit Elements.*

t_2 = 1924 Oct. 28·47775 G.M.A.T.

x + 1·203332	$\dot{x}$ − ·0005799	$1/k$ = 3379·3806	
y + ·226105	$\dot{y}$ + ·0186772		
z + ·333897	$\dot{z}$ + ·0005941	$\sqrt{\frac{1}{k}}$ = 58·13244	
x^2 1·448008	$x\dot{x}$ − ·0006978	$\dot{x}^2$	·000000336
y^2 51123	$y\dot{y}$ + ·0042230	$\dot{y}^2$	348838
z^2 111487	$z\dot{z}$ + ·0001984	$\dot{z}^2$	353
r^2 1·610618	$r\dot{r}$ + ·0037236	c^2	·000349527 × $1/k$
r 1·269101	$(r\dot{r})^2$ ·0000138652	G^2	1·181185
$1/r$ 0·787959		$2/r$	1·575918
		$1/a$	0·394733
		a	2·533358
$(rr)^2/k$ 0·046856	Take $\sqrt{\ }$ + 0·216462	$\dot{a}^2$	6·417903
G^2r^2 1·902438	$\sqrt{p}$ + 1·362198	$\sqrt{a}$	1·591652
p 1·855582	Multiply and get	$a^{3/2}$	4·032224
r 1·269101		$(\sqrt{k})°$	0°·9856077
eaX + 0·586481	eaY + 0·294865	$\mu°$	0°·244433
X + 0·447571	Y + 0·225025		
		$p/a = 1 - e^2$	0·732459
Check		e^2	0·267541
X^2 ·200320		e	0·517244
Y^2 ·050636		ea	1·310364
·250956		$1/ear$	0·601328
$(r/a)^2$ ·250957 (suffices).			

With e = 0·517244 and the U.O. Tables compute

M.	X.		Y.	
6°	0·45970		0·18272	
	− 811		+ 2927	
7°	·45159	− 109	·21199	− 55
	− 920		+ 2872	
8°	·44239	− 107	·24071	− 54
	− 1027		+ 2818	
9°	·43212		·26889,	

and

M = 7°·4517 gives X = 0·44757 Y = 0·22503.

$-r\dot{r}/ear = X = -0{\cdot}00223910$

$\sqrt{(pk)} = {\cdot}0234327$

$\dfrac{a}{\sqrt{(pk)}} = 108{\cdot}1122$

$\dfrac{1}{r} - \dfrac{1}{a} = {\cdot}393226$

$\dfrac{1-e^2}{a-p} = 1{\cdot}080680$

$k = 0{\cdot}0002959122$

$0{\cdot}000319786$

root is $0{\cdot}0178826$

$\dot{Y} + 0{\cdot}00703190.$

Checks

$X\dot{Y} + {\cdot}00314728$	$X\dot{X} - {\cdot}00100216$
$\dot{X}Y - 50385$	$Y\dot{Y} + 158235$
$+ {\cdot}00365113$	$+ {\cdot}00058019$
$\sqrt{(pk)}/a^2 + {\cdot}00365114$	$r\dot{r}/a^2 + {\cdot}00058019$

$$P_x = (x\dot{Y} - \dot{x}Y)\frac{a}{\sqrt{(pk)}}$$
$$Q_x = (\dot{x}X - x\dot{X}) \quad ,,$$

$x\dot{Y} + {\cdot}00846171$	$y\dot{Y} + {\cdot}00158995$	$z\dot{Y} + {\cdot}00234793$
$-\dot{x}Y + 13049$	$-\dot{y}Y - 420284$	$-\dot{z}Y - 13369$
$+ {\cdot}00859220$	$- {\cdot}00261289$	$+ {\cdot}00221424$
$P_x + {\cdot}928922$	$- {\cdot}282485$	$+ {\cdot}239386$
$\dot{x}X - {\cdot}00025955$	$\dot{y}X + {\cdot}00835937$	$\dot{z}X + {\cdot}00026590$
$-x\dot{X} + 269439$	$-y\dot{X} + 50627$	$-z\dot{X} + 74763$
$+ {\cdot}00243484$	$+ {\cdot}00886564$	$+ {\cdot}00101353$
$Q_x + {\cdot}263236$	$+ {\cdot}958484$	$+ {\cdot}109581$

P^2.	PQ.	Q^2.
·862896	+ ·244526	·069293
·079798	− ·270757	·918692
·057306	+ ·026231	·012006
1·000000	− ·000000	0·999991 Discrepancy not searched for.

Multiplying by a and collecting the other elements.

Epoch 1924 Oct. 28·47775 G.M.A.T.

M	7°·4517
μ	0°·244433
e	0·517241.

$$x = +2{\cdot}353292X + 0{\cdot}666871Y$$
$$y = -0{\cdot}715636X + 2{\cdot}428183Y$$
$$z = +0{\cdot}606450X + 0{\cdot}277608Y.$$

As a final partial check, by substituting the values of X and Y we get

+ 1·053266	− 0·320298	+ 0·271429
+ 0·150063	+ 0·546402	+ 0·062469
x + 1·203329	y + 0·226104	z + 0·333898

as against

+ 1·203332	+ 0·226105	+ 0·333897.

Similarly $\dot{x}$, etc., should be reproduced by using $\dot{\mathrm{X}}$, etc.

A comparison with Merton's results from the same observations and with Stracke's based on seven normal places shows :

		Merton.	Stracke.
a	2·53336	2·52271	2·66745
μ	0°·244433	0°·245983	0°·226235
e	0·517244	0·515412	0·539347.

Example (2).—*Comet* 1925*c* (*Orkisz*).

I. *The data given by observation.*

In this case the data were deduced from about ten observations which had been corrected for parallax but not for light-time (Merton, p. 714).

			R.A.	Dec.
t_1	1925 April	5·0000	336° 39′ 42″·9	+ 16° 30′ 8″·1
t_2	,,	9·5000	337 47 54·4	+ 21 17 30·8
t_3	,,	14·0000	339 3 16·3	+ 26 24 27·5.

$$\tau_3 = \tau_1 = 4{\cdot}50000 \qquad \tau_3^2 = 20{\cdot}25000$$
$$\tau_2 = 9{\cdot}00000 \qquad \tau_2^2 = 81{\cdot}00000.$$

$$c_1^0 = c_3^0 = {\cdot}500000.$$

$$\nu_1^0 = \nu_3^0 = c_1^0 \times \frac{k}{6} \times (\tau_2^2 - \tau_1^2) = {\cdot}0001498.$$

$$\frac{k}{6} = {\cdot}0000493187.$$

II. *Solar Co-ordinates.*

In this case the correction for parallax has been applied to the comet's places.

(1)	(2)	(3)
X + ·967846	+ ·946596	+ ·919738
Y + ·232994	+ ·301141	+ ·367508
Z + ·101065	+ ·130626	+ ·159412.

III. *Comet's Direction-Cosines and the Triangle Sun-Earth-Planet.*

$$l = \mathrm{C}\delta\mathrm{C}\alpha \qquad \mathrm{RC}\psi = l\mathrm{X} + m\mathrm{Y} + n\mathrm{Z}$$
$$m = \mathrm{C}\delta\mathrm{S}\alpha$$
$$n = \mathrm{S}\delta \qquad \mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2.$$

	(1)	(2)	(3)
l	+ ·880362	+ ·862664	+ ·836469
m	− ·379838	− ·352074	− ·320177
n	+ ·284053	+ ·363119	+ ·444754.
$2\mathrm{RC}\psi$	+ 1·584525	+ 1·516006	+ 1·445128
R^2	1·001226	1·003793	1·006392.

N.B.—In I., II., and III. the figures are taken without change from Merton, p. 714, save only $2\mathrm{R}_2\mathrm{C}\psi_2$ and $\mathrm{R}_2{}^2$.

IV. *The* 10 *Determinants.*

$$\begin{aligned}
\mathrm{A} &= l_2(m_3n_1 - m_1n_3) + m_2(l_1n_3 - l_3n_1) + n_2(l_3m_1 - l_1m_3)\\
\mathrm{B}_i &= \mathrm{X}_i(m_2n_3 - m_3n_2) + \mathrm{Y}_i(l_3n_2 - l_2n_3) + \mathrm{Z}_i(l_2m_3 - l_3m_2)\\
\mathrm{C}_i &= \mathrm{X}_i(m_3n_1 - m_1n_3) + \mathrm{Y}_i(l_1n_3 - l_3n_1) + \mathrm{Z}_i(l_3m_1 - l_1m_3)\\
\mathrm{D}_i &= \mathrm{X}_i(m_1n_2 - m_2n_1) + \mathrm{Y}_i(l_2n_1 - l_1n_2) + \mathrm{Z}_i(l_1m_2 - l_2m_1)
\end{aligned}$$

$$i = 1, 2, 3.$$

$m_2n_3 - m_3n_2$ − ·0403239	$l_3n_2 - l_2n_3$ − ·0799355	$l_2m_3 - l_3m_2$ + ·0182938
$m_3n_1 - m_1n_3$ + ·0779873	$l_1n_3 - l_3n_1$ + ·1539430	$l_3m_1 - l_1m_3$ − ·0358510
$m_1n_2 - m_2n_1$ − ·0379187	$l_2n_1 - l_1n_2$ − ·0746339	$l_1m_2 - l_2m_1$ + ·0177200.

The three constituents of A are :

+ ·0672768	Check-equation :
− ·0541993	$\mathrm{A} = \mathrm{C}\delta_1\mathrm{C}\delta_2\mathrm{C}\delta_3[\mathrm{T}\delta_1\mathrm{S}(\alpha_3 - \alpha_2) - \mathrm{T}\delta_2\mathrm{S}(\alpha_3 - \alpha_1)$
− ·0130182	$+ \mathrm{T}\delta_3\mathrm{S}(\alpha_2 - \alpha)$
A + ·0000593	gives
1/A + 16863·4	+ ·0000592,75.

	(1)	(2)	(3)
	− ·0390273	− ·0381704	− ·0370874
	− ·0186245	− ·0240719	− ·0293769
	+ ·0018489	+ ·0023896	+ ·0029163
B_1	− ·0558029	B_2 − ·0598527	B_3 − ·0635480
	+ ·0754797	+ ·0738225	+ ·0717279
	+ ·0358678	+ ·0463585	+ ·0565753
	− ·0036233	− ·0046831	− ·0057151
C_1	+ ·1077242	C_2 + ·1154979	C_3 + ·1225881
	− ·0366995	− ·0358937	− ·0348753
	− ·0173893	− ·0224753	− ·0274286
	+ ·0017909	+ ·0023147	+ ·0028248
D_1	− ·0522979	D_2 − ·0560543	D_3 − ·0594791

V. *Equations for* ρ_2 *and* r_2.

$$\begin{array}{rl} & -C_1 - \cdot 1077242 \\ & -C_3 - \cdot 1225881 \\ \hline & -\cdot 2303123 \qquad \times \nu_1^0 = \nu_3^0 - \cdot 0003450{,}1 \\ \hline \times c_1^0 = c_3^0 & -\cdot 1151561{,}5 \\ & C_2 + \cdot 1154979 \\ \hline & +\cdot 0003417{,}5. \end{array}$$

These multiplied by 1/A give

$$\rho_2 = 5\cdot 7631 - 5\cdot 8180/r_2^3 \qquad [0\cdot 76478]$$

with $r_2^2 = \rho_2^2 - 1\cdot 516006\rho_2 + 1\cdot 003793.$

Trials with 1·6 and 1·7 indicate $\rho_2 = 1\cdot 672$, then

$$\begin{array}{rll} \rho_2 = & 1\cdot 6720 & \\ \hline \rho_2^2 & 2\cdot 79558 & \\ 2\rho_2 RC\psi & -2\cdot 53475 & \\ R_2^2 & 1\cdot 00379 & \\ \hline r_2^2 & 1\cdot 26462 & \\ \hline \text{logs} & 0\cdot 10196 & \\ ,, & 0\cdot 05098 & \\ \hline \log r_2^3 & 0\cdot 15294 & 1/r_2^3 \quad 0\cdot 70319 \\ & 0\cdot 76478 & \\ \hline & \left\{\begin{array}{l} 0\cdot 61184 \\ \hline 4\cdot 0911 \end{array}\right. & \\ & 5\cdot 7631 & \\ \hline \rho_2 & 1\cdot 6720. & \end{array}$$

VI. *Derivation of* ρ_1 *and* ρ_3, *etc.*

$$\begin{array}{rl} \nu/r_2^3 = & \cdot 001054 \\ c_0 = & \cdot 500000 \\ \hline c_1 \quad c_3 = & \cdot 501054 \end{array}$$

$$\begin{array}{llll} c_1B_1 + c_3B_3 - \cdot 598011{,}3 & -c_1C_1 - c_3C_3 - \cdot 1153987{,}5 & c_1D_1 + c_3D_3 - \cdot 560063 \\ -B_2 \quad +\cdot 598527 & C_2 \quad +\cdot 1154979 & -D_2 \quad -\cdot 560543 \\ \hline \Sigma B \quad 515{,}7 & \Sigma C + 991{,}5 & \Sigma D + 480 \end{array}$$

$$\Sigma C/A = 1{\cdot}6720 = \rho_2$$
$$\rho_2/c_1 = \rho_2/c_3 = 3{\cdot}3370$$

$\Sigma B/\Sigma C = 0{\cdot}5197$ $\Sigma D/\Sigma C = 0{\cdot}4841,$

giving

ρ_1 1·73562 ρ_2 1·67200 ρ_3 1·61547.

Then

	(1)	(2)	(3)
ρ^2	3·0052	2·7956	2·6098
$2R\mathrm{C}\psi\rho$	− 2·7468	− 2·5348	− 2·3346
R^2	1·0012	1·0038	1·0064
r^2	1·2596	1·2646	1·2818
$1/r^3$	0·7073	0·7032	0·6893

VII. *Corrections for Light-times and adjusted values of c_1 and c_3.*

$\cdot 005770 \times \rho$

t_1	− ·01001	t_1' 4·98999
t_2	− ·00965	t_2' 9·49035
t_3	− ·00932	t_3' 13·99068

$\tau_3 = 4{\cdot}50036$
$\tau_1 = 4{\cdot}50033$
$\tau_2 = 9{\cdot}00069.$

$c_1^0 = \tau_1/\tau_2$ ·499998
$c_3^0 = \tau_3/\tau_2$ ·500002
} Always sum to unity.

$\tau_2\tau_3$	40·50634	$\tau_1\tau_3$	20·25310	$\tau_1\tau_2$	40·50608
τ_1^2	20·25297	τ_2^2	81·01242	τ_3^2	20·25324
	20·25337		101·26552		20·25284

$$\times \frac{k}{12} = \cdot 0000246593.$$

	·00049943	·00249714	·00049942
$\times 1/r_i^3$			
	·0003532	·0017560	·0003443
	17591	square + 31	17591
	1·002112,3	Add unity	1·002103,4
$\times c_1^0$			$\times c_3^0$
c_1	·501054		c_3 ·501054.

The values of c_1 and c_3 are unchanged. The term arising from $\dot{r} = \pm 4$ in the sixth decimal place, but it is already included automatically.

The values of the ρ's will not be changed, and no revision is necessary.

VIII. *Heliocentric Co-ordinates.*

	ρ_1 1·73562.	ρ_2 1·67200.	ρ_3 1·61547.	$c_1x_1+c_3x_3$.	Residual.
ρl	+ 1·527974	+ 1·442374	+ 1·351291		
X	+ ·967846	·946596	+ ·919738		
x	+ ·560128	+ ·495778	+ ·431553	+ ·496886	− 1108
ρm	− ·659254	− ·588668	− ·517236		
Y	+ ·232994	+ ·301141	+ ·367508		
y	− ·892248	− ·889809	− ·884744	− ·890369	+ 560
ρn	+ ·493008	+ ·607135	+ ·718487		
Z	+ ·101065	+ ·130626	+ ·159412		
z	+ ·391943	+ ·476509	+ ·559075	+ ·476511	− 2

The corrections to ρ_1 and ρ_3 to bring them into dynamical agreement with ρ_2 will be found from equations of the type

$$c_1l_1\Delta\rho_1 + c_3l_3\Delta\rho_3 = \text{Residual},$$

or in this case

$$\begin{aligned} &+ c_1\cdot 8804\Delta\rho_1 + c_3\cdot 8365\Delta\rho_3 = -1108 \\ &- c_1\cdot 3798 \;,, \quad - c_3\cdot 3202 \;,, \quad = +\;560 \\ &+ c_1\cdot 2841 \;,, \quad + c_3\cdot 4448 \;,, \quad = -\;\;2, \end{aligned}$$

or

$$\begin{aligned} &\cdot 4411\Delta\rho_1 + \cdot 4191\Delta\rho_3 = -\cdot 001108 \\ &\cdot 1903 \;,, \quad + \cdot 1604 \;,, \quad = -\;\;560 \\ &\cdot 1423 \;,, \quad + \cdot 2228 \;,, \quad = -\;\;\;2. \end{aligned}$$

A few trials give

$$\begin{aligned} \Delta\rho_1 &= -\cdot 006391 \\ \Delta\rho_3 &= +\cdot 004082. \end{aligned}$$

Substituting, we have

$$\begin{aligned} &-2819 + 1711 = -1108 \text{ as against } -1108 \\ &-1216 + \;655 = -\;561 \qquad\qquad -\;560 \\ &-\;909 + \;909 = \;\;\;0 \qquad\qquad -\;\;\;2, \end{aligned}$$

which is good enough.

We have therefore

$$\begin{aligned} \rho_1 &= 1\cdot 735620 - 6391 = 1\cdot 729229 \\ \rho_3 &= 1\cdot 615470 + 4082 = 1\cdot 619552. \end{aligned}$$

Revising the calculations for x_1, x_3, etc., we have

	x.	y.	z.	$c_1x_1 + c_3x_3$, etc.	
(1)	+ ·554502	− ·889821	+ ·390128	·495777	
(2)	·495778	− ·889809	+ ·476509	·889808	x_2, etc.
(3)	·434967	− ·886051	+ ·560890	·476511	

IX. *The Velocities.*
From VI.

$$1/r_1^3 = 0{\cdot}7073$$
$$1/r_3^3 = 0{\cdot}6893$$
$$0{\cdot}0180$$

$1/r_2^3 =$	0·7032	0·7032	0·7032	0·7032
$\frac{1}{6}$th +	30	$\frac{1}{4}$ + 45	− 30	− 45
	0·7062	0·7077	0·7002	0·6987

$$\times \frac{k}{2} = \cdot 0001479\,56.$$

	·0001045	1047			·0001036	1034		
$\times \tau_3^2$				$\times \tau_1^2$				
	·002116	2120			·002098	2094		
		707				698		
f_1	·997884	·999293		f_3	·997902	·999302		
			$\times \tau_3$					$\times \tau_1$
		g_1 4·49718				g_3 4·49719.		

Check

$$f_1 g_3 \quad 4{\cdot}48767$$
$$f_3 g_1 \quad 4{\cdot}48774$$
$$g_2 \quad 8{\cdot}97541$$

$c_1 = g_3/g_2 = 0{\cdot}501057 \quad c_3 = g_1/g_2 = 0{\cdot}501056$

Compare the values used in VI. The small differences need not be considered.

x_1 + ·554502	x_3 + ·434967	− ·0132913
$f_1 x_2$ + ·494729	$f_3 x_2$ + ·494738	132907
$g_1 \dot{x}$ − 59773	$g_2 \dot{x}$ − 59771	$\dot{x}$ − ·0132910
y_1 − ·889821	y_3 − ·886051	+ ·0004214
$f_1 y_2$ − ·887926	$f_3 y_2$ − ·887942	4205
$g_1 \dot{y}$ + 1895	+ 1891	$\dot{y}$ + ·0004210
z_1 + ·390128	z_3 + ·560890	+ ·0189837
$f_1 z_2$ + ·475501	$f_3 z_2$ + ·475509	189854
$g_1 \dot{z}$ + 85373	$g_2 \dot{z}$ + 85381	$\dot{z}$ + ·0189846

In the present example, the time-intervals are virtually equal and this facilitates the application of the Cowell process. It will therefore be exemplified.

Knowing the co-ordinates for each of the three dates, we can compute the accelerations, and from them the second differences, thus:

$$\Delta^2 x = \ddot{x} - \tfrac{1}{12}\Delta^2\ddot{x} + \tfrac{1}{240}\Delta^4\ddot{x} - \text{, etc.}$$

In this

$$\ddot{x} = -k\tau^2/r^3$$

$k = \cdot 0002959122$
$k\tau^2 = \cdot 0059931$ $\qquad \tau^2 = 20{\cdot}2531$
(Mean of τ_1^2 and τ_3^2, VII.).

A recomputation of $1/r^3$ with the revised data give respectively

	·7143	·7032	·6834
and			
$k\tau^2/r^3$	·004281	4214	4096.

We now form a difference table of the co-ordinates and a similar table for the accelerations :—

	x.	Δx.	$\Delta^2 x$.		$\ddot{x}$.	$\Delta^2\ddot{x}$.	$\Delta^2 x$.
t_1	+ ·554502				− ·002374		
		− 58724				+ 285	
t_2	·495778		− 2087		− · 2089	+ 22 ∴	− 2087
		− 60811				+ 307	
t_3	·434967				− · 1782		

	y				$\ddot{y}$		
t_1	− ·889821				+ ·003809		
		+ 12				+ 59	
t_2	− ·889809		+ 3746		3750	+ 62 ∴	+ 3745
		+ 3758				+ 121	
t_3	− ·886051				3629		

	z				$\ddot{z}$		
t_1	+ ·390128				− ·001670		
		+ 86381				− 338	
t_2	·476509		− 2000		− 2008	+ 49 ∴	− 2004
		+ 84381				− 289	
t_3	·560890				− 2297		

It will be seen that the second differences derived in the two rather different ways agree very well. The largest difference is in the z, and it can be adjusted as follows :—

$z_3 - z_1 = + \cdot 170762$
$\Delta^2 z_2 = \quad 2004$

Take this + and −

·172766	or	Δ = 86383	adjusted first
·168758		= 84379	differences,

giving

	z.	Δz.	$\Delta^2 z$.
t_1	+ ·390127		
		+ 86383	
t_2	+ ·476510		− 2004
		+ 84379	
t_3	+ ·560889		

From the way that x, y, and z have been built up it is obvious that they must represent the observations exactly; save for the minute change just made in the z co-ordinate. The greatest effect will be in the third position, so we work out the geocentric place from it:

	logs.	
$y + Y - 0{\cdot}518543$	$9{\cdot}7147848n$	
$x + X + 1{\cdot}354705$	$0{\cdot}1318447$ —	
	(296897)—	
$T\alpha$	$9{\cdot}5829401n$	$\alpha = 339°\ 3'\ 16''{\cdot}4$
$z + Z + {\cdot}720301$	$9{\cdot}8575140$	
	$0{\cdot}1615344$ ←	
$T\delta$	$9{\cdot}6959796$	$\delta = +26°\ 24'\ 27''{\cdot}3.$

The differences O − C are − 0″·1 and + 0″·2, and for positions 1 and 2 they will be smaller. Thus the gravitational process represents the observations exactly, whereas the assumption of a parabola gives for the middle place

$$\left.\begin{matrix}\Delta\alpha\mathrm{C}\delta & +\ 4'' \\ \Delta\delta & -\ 2''\end{matrix}\right\}\text{Merton, p. 716,}$$

which, it is remarked by the author, may be considered satisfactory for a cometary orbit. Perhaps "parabolic" would be a better statement. Nevertheless, it is quite probable that the path just found is considerably in error.

It is now easy to compute a heliocentric ephemeris, which for a few dates can be done at sight with only simple additions and subtractions.

If we look at the table of co-ordinates, we see for example that the x table can be continued as follows:—

	x.	Δx.	$\Delta^2 x$.	$\ddot{x}$.	$\Delta\ddot{x}$.	$\Delta^2\ddot{x}$.
t_0						
t_1	+ ·554502		− 2372 :	− 2374		
		− 58724			+ 285	
t_2	·495778		− 2087	− 2089		+ 22
		− 60811			+ 307	
t_3	·434967		− 1780 :	− 1782		
t_4						

For the moment there is a little guess-work in the figures marked : but it is quite unlikely that they will be in error by more than a unit

in the last place. Admitting them for the moment, we can find x for t_0 and t_4. Then $\ddot{x}$ can be computed exactly and the adjustments made, if there are any, just the simple but exact Cowell process. The similar tables for y and z are omitted, but for t_0 we get

$x + \cdot610854$ $\quad y - \cdot886029$ $\quad z + \cdot302078$ $\quad r^2$ 1·2495.

So we have

	r^2.	Δ.	Δ^2.	Δ^3.
t_0	1·2495			
		+ 19		
t_1	1·2514		+ 113	
		+ 132		+ 8
t_2	1·2646		+ 121	
		+ 253		
t_3	1·2889			

This table shows that r was a minimum at the time $t_1 - 4^{d}\cdot5 \times \cdot668$ for which $r^2 = 1\cdot2489$, or

Perihelion T 1925 April 1·499
$q = 1\cdot11745$.

X. *The Orbit.*

The data are

t_2 1925 April 9·50 G.C.T.

$x + \cdot495778$	$\dot{x} - \cdot059817$	k	·0002959122
$y - \cdot889809$	$\dot{y} + \cdot001870$	$k\tau^2$	·00599314.
$z + \cdot476510$	$\dot{z} + \cdot085407$		

N.B.—These velocities have been derived directly from the 1st and 3rd differences by the formula

$$\dot{x} = \tfrac{1}{2}(\Delta_1 + \Delta_2) - \tfrac{1}{12}(\Delta_1^3 + \Delta_2^3) + \text{, etc.,}$$

assuming that in the present case Δ_1^3 and Δ_2^3 may be confused with $\Delta_1\ddot{x}$ and $\Delta_2\ddot{x}$, etc. The details are

$$\begin{aligned} \dot{x} &= -59768 - \tfrac{1}{12}(\ \ 592) = -\cdot059817 \\ \dot{y} &= +\ 1885 - \tfrac{1}{12}(\ \ 180) = +\cdot001870 \\ \dot{z} &= +85381 - \tfrac{1}{12}(-\ 314) = +\cdot085407. \end{aligned}$$

Then

x^2	·245796	$x\dot{x}$	$-\cdot029656$	$\dot{x}^2$	·0035781
y^2	·791760	$y\dot{y}$	$-\cdot001664$	$\dot{y}^2$	·0000035
z^2	·227062	$z\dot{z}$	$+\cdot040697$	$\dot{z}^2$	·0072944
r^2	1·264618	$r\dot{r}$	$+\cdot009377$	c^2	·0108760 } $\times 1/k\tau^2$
r	1·124552	$(r\dot{r})^2$	·000087928	G^2	1·814741 }
$2/r$	1·778485			$\frac{2}{r}$	1·778485
$1/r$	0·889242			$1/a$	$-0\cdot036256$
				a	27·5816

$(r\dot{r})^2/k\tau^2$ ·0146714 $1/a^2$ ·0013145

G^2r^2 2·2949541

p 2·2802827

p/a − 0·0826739 = $1 - e^2$

1·0826739 = e^2

1·040516 = e q = 1·117496.

The orbit is hyperbolic, so that it is well-nigh useless to compute the hyperbolic elements and an ephemeris by their means. The process given in IX. is much simpler.

If c^2 = ·0106587 instead of ·0108760, then the orbit would be parabolic, but, as stated earlier, the orbit is actually hyperbolic (Goryatschew, *M.N.*, 1928 June), but with a much smaller excentricity than given by the present solution.

The process of finding the elements still continues unchanged—it being the same for all conic sections.

p 2·280283 $(er)^2$ 1·3691690

r 1·124552 $(p-r)^2$ 1·3357141

$p - r$ 1·155731 = eaX 0·0334549 = $(ea\mathrm{Y})^2$

− 0·182907 = eaY

$1/ea$ − 0·0348442 $1/ear$ − 0·0309850

X − ·0402706 Y − ·00637326 Y same sign as $\frac{r\dot{r}}{a}$

$$\dot{\mathrm{X}} = -r\dot{r}/ear = +\cdot 00029055.$$

Multiply these three,

$$\frac{1}{r} - \frac{1}{a} = +\cdot 925498$$

$$\frac{\sqrt{(e^2-1)}}{e} = +\cdot 2763347$$

$$\sqrt{\frac{k\tau^2}{a}} = -\cdot 01474067 \qquad \frac{k\tau^2}{a} \cdot 0002172873,$$

and get $\dot{\mathrm{Y}}$ − ·0037699.

Checks

$\mathrm{X}\dot{\mathrm{X}}$ − ·0000117006 $\mathrm{X}\dot{\mathrm{Y}}$ + ·0001518062

$\mathrm{Y}\dot{\mathrm{Y}}$ + 240251 $\dot{\mathrm{X}}\mathrm{Y}$ − 18518

+ ·0000123245 + ·0001536620

$r\dot{r}/a^2$ + ·0000123245 $\frac{\sqrt{(pk\tau^2)}}{a^2}$ + ·000153662.

We may also compute Y as follows:—

$\sqrt{(k\tau^2)}$ ·0774155 $r\dot{r}/\sqrt{k\tau^2}$ ·1211256

$\sqrt{p}$ 1·510060

$1/ea$ − ·0348442

Multiply out and get Y = − ·00637325.

From

$$P_x = (x\dot{Y} - \dot{x}Y) \cdot \frac{a}{\sqrt{(pk\tau^2)}}$$
$$Q_x = (\dot{x}X - x\dot{X}) \quad ,,$$

$$\sqrt{(pk\tau^2)} = \cdot 1169020 \qquad a/\sqrt{(pk\tau^2)} = -235\cdot 9378.$$

$P_x + \cdot 530921$	$Q_x - \cdot 534357$	$P_xQ_x - \cdot 283701$
$P_y - \cdot 794263$	$Q_y - \cdot 043229$	$P_yQ_y + \quad 34335$
$P_z + \cdot 295413$	$Q_z + \cdot 844148$	$P_zQ_z + \cdot 249372$
		$+ \cdot 000006$

$$\Sigma P^2 = 1\cdot 000000 \qquad \Sigma Q^2 = 0\cdot 999992.$$

It has not been thought worth while to bring ΣQ^2 to unity and the sum of PQ to nil.

Multiplying by a we have finally

$$x = -14\cdot 6437X + 14\cdot 7384Y$$
$$y = +21\cdot 9070X + 1\cdot 1923Y$$
$$z = -\ 8\cdot 1480X - 23\cdot 2830Y$$

T 1925 April 1·499
$q = 1\cdot 117496$
$e = 1\cdot 040516$
$a = -27\cdot 5816.$

So far as at once comparable, we have

	This paper.	Merton's Parabola.	Goryatschew's Definitive Elements.
T 1925	April 1·1499	1·2468	1·51061
q	1·1175	1·1095	1·1096
e	1·0405	1·0000	1·0006408.

The P_x, Q_x, etc., Quantities.

These very convenient quantities were explicitly introduced to astronomers by Adams (see *B.A.J.*, **32,** 232), but forms which do not differ greatly were used earlier by Thiele (see *U.O. Circ.*, **68,** 355). They have the great advantage of separating the dynamical and positional elements—and scale—when written

$$x = aP_xX + aQ_xY$$
$$y = aP_yX + aQ_yY$$
$$z = aP_zX + aQ_zY,$$

in which

$$X = \frac{r}{a}Cv$$

$$Y = \frac{r}{a}Sv,$$

and as already mentioned, X and Y can be taken from the U.O. tables. In Dr. Stracke's tables, mentioned on p. 427, he virtually adopts all the above, but unfortunately uses a different and much inferior notation (m and n, C and S) as m and n are also used as precessional numbers, mass, mean motion, etc., and C and S seem to be weak. Still we are

grateful for Dr. Stracke's recognition of the new constants and his valuable elliptic co-ordinate table.

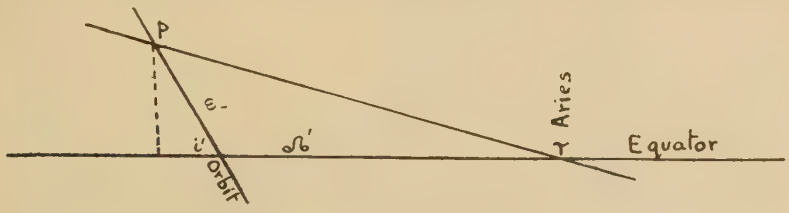
The P, Q quantities are easily derived from the Gaussian equatorial co-ordinates or directly from Ω, ω, and i and ϵ (see Merton, *Mem.*, *R.A.S.*, **64**, 103, and Kurt Laves, *A.N.*, 5589).

Their geometrical significance is shown in the diagram (Ω', ω', and i' are referred to the equator).

Let α and δ be the right ascension and declination of the perihelion-point. Then

$$\left.\begin{aligned} P_x &= C\delta C\alpha \\ P_y &= C\delta S\alpha \\ P_z &= S\delta \end{aligned}\right\}$$ The Q's have a similar significance, but for a point advanced 90° on the orbit, that is for $\omega + 90°$.

These equatorial constants will generally be referred to a normal equinox such as that of 1925, and it may become necessary to refer



them to another normal equinox such as that of 1950. This can be done in several ways ; since

$$P_x = C\delta C\alpha,$$

the change due to precession can be applied to α and δ and the new value of P_x then derived.

This trigonometrical solution can be avoided and the whole process generalised by adopting a method used by Banachiewicz.

If the equations (14) given by Newcomb on p. 267 of his *Compendium* are expanded and

$$\left.\begin{aligned} C\delta C\alpha &\text{ is replaced by } P_x \text{ or } Q_x \\ C\delta S\alpha &\quad ,, \quad ,, \quad P_y ,, Q_y \\ S\delta &\quad ,, \quad ,, \quad P_z ,, Q_z \end{aligned}\right\}$$, as the case may be,

using the same symbols accented for 1950, and abridging as follows :—

$$\begin{aligned} a_{1,1} &= -SzSz_0 + CzCz_0C\theta \\ a_{1,2} &= -SzCz_0 - CzSz_0C\theta \\ a_{1,3} &= -CzS\theta \\ a_{2,1} &= CzSz_0 + SzCz_0C\theta \\ a_{2,2} &= CzCz_0 - SzSz_0C\theta \\ a_{2,3} &= -SzS\theta \\ a_{3,1} &= Cz_0S\theta \\ a_{3,2} &= -Sz_0S\theta \\ a_{3,3} &= C\theta, \end{aligned}$$

then
$$\left.\begin{aligned} P_x' &= a_{1,1}P_x + a_{1,2}P_y + a_{1,3}P_z \\ P_y' &= a_{2,1}\,,, \quad a_{2,2}\,,, \quad a_{2,3}\,,, \\ P_z' &= a_{3,1}\,,, \quad a_{3,2}\,,, \quad a_{3,3}\,,, \end{aligned}\right\}$$ For the Q quantities replace the P's by Q's.

The values of z, z_0, and θ are given by Newcomb on p. 414 of his *Compendium*, and for the interval 1925–1950 are:

		Sine.	Cosine.
z	576″·22	0·0027,9359	0·9999,9616
z_0	576·17	0·0027,9335	0·9999,9616
θ	501·09	0·0024,2935	0·9999,9705.

These values give

		Banachiewicz.
$a_{1,1}$	= 1 − 0·0000,1855	− 1856
$a_{1,2}$	= − 0·0055,8641	− 55,8690
$a_{1,3}$	= − 0·0024,2934	− 24,2934
$a_{2,1}$	= + 0·0055,8691	+ 55,8691
$a_{2,2}$	= 1 − 0·0000,1560	− 1561
$a_{2,3}$	= − 0·0000,0679	− 679
$a_{3,1}$	= + 0·0024,2934	+ 24,2934
$a_{3,2}$	= − 0·0000,0679	− 679
$a_{2,3}$	= 1 − 0·0000,0295	− 295.

The $a_{i,j}$ quantities have been determined by Th. Banachiewicz by an entirely different analysis which will be found in his paper, *Expressions analytiques pour le calcul de la précession en coordonnées orthogonales*, Cracowie, 1924, and his figures for 1925 to 1950, which are repeated above in the last column, will be found on p. 55 of *Rocznik Astron. Obser.*, Krakowskeigs, **3,** 1925. They are founded upon the Newcomb-Andoyer data given in the *Bull. Astron.*, 1911, p. 72. (See also B. Numerov's Table V. in No. 13 of the *Bulletin of the Astronomical Institute*, Leningrad, 1926.) The slight discrepancies between the two sets of figures given above have not been investigated.

Until 1950, the precession to date from 1925·0 should be given in the national ephemerides, as is indeed already done in the *B.J.* for 1929 (see pp. 280–286). In this connection reference should be made to item (4) of L. J. Comrie's paper on the use of a normal equinox (*M.N.*, 1926 June, p. 620). In the *N.A.* for 1931, Tables XIII. and XIV. are given for the reduction of star positions from 1931 to the standard equinox of 1950. It should be remarked that these tables, as well as the table on p. 280 of the *B.J.*, do not require the preliminary use of $(\alpha + \alpha')/2$ and $(\delta + \delta')/2$; they are what might be called Ristenpart tables. The orthogonal process also does not require the middle co-ordinates, and it is quite an open question if the x, y, z method will not be found the more convenient. If so, tables will be to hand in the annual international supplements prepared by Th. Banachiewicz (see for example his table "*Reduction to true Place*" for each 4th day of 1925 in *Rocznik*, **3,** 56–7); but the later years have not so far been received here.

Johannesburg:
1928 *October* 28.

A Modification of Gauss's Method for the Determination of Orbits.

By G. Merton, M.A., Ph.D.

The original paper under this title is to be found in volume **85** (pp. 693–731). Dr. Innes in his paper above makes a number of references to it, and it therefore seems a suitable opportunity to publish such errata, amendments, and notes as have been collected. A number of these have come to light as the result of further experience of the method, mostly by other astronomers, to whom I am also indebted for much helpful discussion and correspondence.

P. 694. Equations (5); in equation for ν_1, *for* $\frac{1}{r}$ *read* $\frac{1}{r_2}$.

P. 702. Equation (52)—Note, a simpler, but less exact, form is :

$$c_1{}^0 r_1{}^2 + c_3{}^0 r_3{}^2 - r_2{}^2 = \tau_1 \tau_3 \left(\frac{1}{r_2} - \frac{1}{a} \right).$$

P. 703, line 12. *For* "motion" *read* "curvature." It is the curvature of the apparent path upon which one chiefly relies for the determination of the geocentric distance. The method here used ensures that two of the three observations are exactly represented by the resultant parabolic orbit, and it would seem that, for the representation of the remaining observation, it is best to choose those two equations of (9) which show the greatest accelerations in the direction cosines; namely, choose the two equations showing the largest numerical values (without regard to sign) for

$$(p_3 - p_2) - \frac{\tau_1}{\tau_3}(p_2 - p_1), \quad \text{where} \quad p = l, m, \text{ or } n.$$

P. 705, line 8. *For* "equation" *read* "equations."

P. 706. The last paragraph needs amendment; see *M.N.*, **86**, 148–151.

P. 707, equation (67), last line. *Insert* c_3 after $-\cdot 3973800$.

P. 710, line 11 from foot. *For* ν_1 *read* ν_3.

P. 710, last line. *For* $l_1|$ *read* $|l_1|$. This means the numerical value of l_1 without regard to sign. With regard to this and other examples of eliminations, see note below.

P. 713, line 4 from foot. *After* "accuracy" *add* "whatever the method employed."

P. 714. In V., equations for $c_3\rho_3$ and $\Delta\rho$, drop the last decimal place which, owing to the previous multiplication ($\times$ 9 abt.), has no significance. See also note below.

P. 714, last line. *For* z_2 *read* z_3.

P. 715, line 23 (VI. line 9). *For* $c_3{}^0 r_2$ *read* $c_3{}^0 r_3$.

P. 715, line 7 from foot. The value of $\Delta\rho$ contains a computational error. It should have been $-\cdot 006241$, and the subsequent work requires slight modification. The fact, however, that the resulting elements fit the observations satisfactorily, implies that an accuracy

of only two significant figures in $\Delta\rho$ was justified by the data. See also note below.

Essentially it is this quantity $\Delta\rho$ that is neglected in Olbers's method (first solution), and this gives rise to errors, usually less than 2′, such as a two- or three-figure value of $\Delta\rho$ might be expected to remove.

Pp. 719–720. In connection with formulæ for c_1, c_3, and $\bar{y}$ (ratio sector : triangle), those given by K. Bohlin (*A.N.*, **219**, 318) probably deserve attention for approximate work.

P. 720, line 3. In equation for h_2, *for* κ *read* κ_2.

P. 721. To XI. may be added :

Period, $P = a^{\frac{3}{2}}$ sidereal years.
$= [2{\cdot}562598]a^{\frac{3}{2}}$ mean solar days.

P. 722, line 11 from foot, *for* " motion " *read* " curvature." See erratum to p. 703, above.

P. 722, at foot. Note rough check : $N_0 = -\frac{N_1}{R_2^3}$ nearly.

P. 723, line 5. η is the same as the quantity called ν in Klinkerfues, *Theoret. Astron.*, Table X.

P. 723, line 8. For a first approximation : $s^2 = 2\frac{\tau_2^2}{r}$, roughly.

P. 723, line 14. *After* " equation " *add* " by multiplying out and adding the above squares."

P. 725, line 10 from foot. In check, *for* $qP_y 2Q_y$ *read* $qP_y 2qQ_y$.

P. 726, line 17. *For* $c_2\rho_3$ *read* $c_3\rho_3$.

P. 726, line 12 from foot. The denominator of the equation for η should *read* $(r_2 + r_3)^3$.

P. 728. The value of ΔZ for Arequipa, *for* + 102 *read* + 120.

A word may be said about the desirability of keeping a note of the order of accuracy in the numerical work at each stage of the calculation, in order to avoid meaningless figures and useless work in attempting to improve solutions beyond the possible significant accuracy. My own example, pp. 714–5, is a case in point, since my computational error in $\Delta\rho$ showed that really it had only two significant figures (see remarks to errata to pp. 714 and 715, above). There is little doubt that the computer can save much time by an intelligent regard for such matters ; special considerations, however, such as those on p. 697, must not be forgotten.

A similar matter is involved in the practical process of eliminating unknowns from a number of given equations, such as in IV. and V. (p. 710, etc.). A useful check on the accuracy to which the problem is determinable is obtained if, in eliminations, the computer will *always* proceed so that he uses a multiplying factor of *less* than unity ; that is, chooses the equation with the largest coefficient for the quantity to be eliminated and reduces it to equality with the other equation or equations having the smaller coefficient or coefficients. In this way the decimal places retain their original significance in accuracy, and if the original equations were reliable, say, to one unit in the fifth place, then

the final equation is also reliable to that order. It is thus possible to tell at once to what number of significant figures the problem is determinable, and it enables one to cut out all refinements which do not affect that order of accuracy.

In connection with the $P_x Q_x$ formulæ, a suitable arrangement for obtaining these from the elements ω, Ω, and i, when a calculating machine is availabe, is:

Tabulate $\sin\omega$, $\cos\omega$, $\sin\Omega$, $\cos\Omega$, $\sin i$ and $\cos i$.

Then compute

$$\begin{aligned} \sin\omega\sin\Omega &= (1) & \cos\omega\sin\Omega &= (4)\\ \sin\omega\cos\Omega &= (2) & \cos\omega\cos\Omega &= (5)\\ \sin\omega\sin i &= (3) & \cos\omega\sin i &= (6). \end{aligned}$$

Then

$$\begin{aligned} (5) - (1)\cos i &= (7) = P_x & -(2) - (4)\cos i &= (9) = Q_x\\ (5)\cos i - (1) &= (8) & (2)\cos i + (4) &= (10). \end{aligned}$$

Finally,

$$\begin{aligned} P_y &= (10)\cos\epsilon - (3)\sin\epsilon & Q_y &= (8)\cos\epsilon - (6)\sin\epsilon\\ P_z &= (3)\cos\epsilon + (10)\sin\epsilon & Q_z &= (6)\cos\epsilon + (8)\sin\epsilon. \end{aligned}$$

where ϵ = obliquity of the ecliptic.

Before concluding, two errata to my paper in *Memoirs*, **64**, may be mentioned.

P. 55 Equation (6), *read* $u' = v + \omega - N = u - N$.

P. 84, Table Vb. Star 107 should *read* B.D. + 45°, 2217.

London:
1928 *December* 12.

Alignment Chart connecting Position Angle and Distance of the Apex of the Sun's Way, and Galactic Co-ordinates, with Equatorial Co-ordinates. By J. M. Baldwin, M.A., D.Sc., F.Inst.P., Government Astronomer for Victoria.

1. In the *Publications of the Dominion Astrophysical Observatory*, vol. iv., No. 4, graphs are given for obtaining the position angle and distance of the solar apex and the galactic co-ordinates of any star.*

Taking the position of the apex as R.A. 271°, Dec. + 28°; the position of the ascending node of the galactic plane on the equator and the inclination of the galactic plane to the equator as 280° and 62°; the equations are

$$\begin{aligned} \text{(i)}\ \cos D &= \cos 62^\circ\,.\cos(90^\circ-\delta) + \sin 62^\circ\,.\sin(90^\circ-\delta)\,.\cos(\alpha - 271^\circ),\\ \text{(ii)}\ \cos 62^\circ &= \cos D\,.\cos(90^\circ-\delta) + \sin D\,.\sin(90^\circ-\delta)\,.\cos(360^\circ-\chi),\\ \text{(iii)}\ \cos(90^\circ-b) &= \cos 62^\circ\,.\cos(90^\circ-\delta) + \sin 62^\circ\,.\sin(90^\circ-\delta)\,.\cos(\alpha - 190^\circ),\\ \text{(iv)}\ \cos(90^\circ-\delta) &= \cos 62^\circ\,.\cos(90^\circ-b) + \sin 62^\circ\,.\sin(90^\circ-b)\,.\cos(90^\circ-l), \end{aligned}$$

* See also Smart, *M.N.*, **83**, 465, 1923.

where

α, δ are the R.A. and Dec. of the star;
D, χ are the distance and position angle of the apex from the star; and
l, b are the galactic longitude and latitude.

Each of these equations can be solved readily by an alignment chart. (For the solution of the general spherical triangle see d'Ocagne, *Traité de Nomographie*, pp. 327 and 329.)

Writing equations (i), (iii), and (iv) in the form

$$-\cos a + \sin 62^\circ . \cos c . \cos \mathrm{A} + \cos 62^\circ . \sin c = 0,$$

the corresponding alignment chart is given by

$$u = -m \cos a, \qquad v = m \cos \mathrm{A},$$

$$x = \mathrm{OQ} . \frac{\sin 62^\circ . \cos c - 1}{\sin 62^\circ . \cos c + 1}, \qquad y = -m . \frac{\cos 62^\circ . \sin c}{\sin 62^\circ . \cos c + 1},$$

where u, v are distances measured along parallel lines Pu, Qv respectively; O, the mid-point of PQ, is the origin for (x, y), Oy is parallel to Pu, and Ox falls along PQ; and m has any convenient value.

The values for a, A, and c are given by

	(i)	(iii)	(iv)
a	D	$90^\circ - b$	$90^\circ - \delta$
A	$\alpha - 271^\circ$	$\alpha - 190^\circ$	$90^\circ - l$
c	δ	δ	b

Equation (ii) can be written

$$\begin{aligned} 2 \cos 62^\circ &= \cos \mathrm{D} . \sin \delta . (\overline{1 + \cos \chi} + \overline{1 - \cos \chi}) \\ &\quad + \sin \mathrm{D} . \cos \delta . (\overline{1 + \cos \chi} - \overline{1 - \cos \chi}) \\ &= (1 + \cos \chi) . \sin (\mathrm{D} + \delta) - (1 - \cos \chi) . \sin (\mathrm{D} - \delta) \end{aligned}$$

or

$$-(1 + \cos \chi) . \sin (\mathrm{D} + \delta) + (1 - \cos \chi) . \sin (\mathrm{D} - \delta) + 2 \cos 62^\circ = 0.$$

The alignment chart for this is given by

$$u = -m . \sin (\mathrm{D} + \delta), \qquad v = m . \sin (\mathrm{D} - \delta),$$

$$x = \mathrm{OQ} . \frac{(1 - \cos \chi) - (1 + \cos \chi)}{(1 - \cos \chi) + (1 + \cos \chi)} = -\mathrm{OQ} . \cos \chi,$$

$$y = m . \frac{-2 \cos 62^\circ}{(1 - \cos \chi) + (1 + \cos \chi)} = -m \cos 62^\circ.$$

Thus the same u and v scales as before, with appropriate numbering, will serve for this diagram also, the χ support being the straight line $y = -m \cos 62^\circ$.

2. The points represented by the above equations for u, v, x, y have been plotted on a diagram which is being reproduced separately, the supports being shown by full lines, and the arguments to be used for equations (i), (ii), (iii), or (iv) being given near the support, with the appropriate number. Thus

(i) For the distance D of the solar apex from the star (α, δ), set a straight line at $\alpha - 6^h\ 4^m$ on the v scale and at δ on the curved support. The intersection of this straight line with the u scale gives the value of $D + 90°$ (right-hand numbers).

(ii) For the position angle χ of the solar apex, set the straight line at $D + \delta$ on the u scale and at $D - \delta$ on the v scale. The intersection with the horizontal scale gives the value of χ, which must lie between 0° and 180° when $\alpha - 6^h\ 4^m$ lies between 0^h and 12^h.

(iii) For the galactic latitude b of the star (α, δ), set the straight line at $\alpha - 0^h\ 40^m$ on the v scale and at δ on the curved support. The intersection with the u scale gives the value of b (left-hand numbers).

(iv) For the galactic longitude l, set the straight line at δ on the u scale and at b on the curved support. The intersection with the v scale gives the value of l, which must lie between 90° and 270° when $\alpha - 0^h\ 40^m$ lies between 0^h and 12^h.

For finding (α, δ) when (l, b) are given, first find δ from l and b (iv), and then find $\alpha - 0^h\ 40^m$ from b and δ (iii), noting that $\alpha - 0^h\ 40^m$ lies between 0^h and 12^h when l lies between 90° and 270°.

3. Both the u and v scales become very congested near the ends, so that the error may be rather large when the reading off has to be done in these portions of the scales. If the quantity to be read off is near the end of the v support, the range in the possible values of u is large. Consequently, although the two ends of the v scale have been expanded and plotted on the two dotted straight supports A and B, the same u scale as before is to be used in connection with them. The corresponding curved supports are shown by dotted lines and marked A and B respectively.

If the quantity to be read off is near the end of the u support, it can only arise from a reading near the other end of the v support. Consequently the end portions of both scales can be expanded. The expanded v scales A and B have been used, the expanded u scales (dotted supports) being placed near them, so as to form two small square diagrams. The curved supports (also dotted) lie between the u and v scales.

4. In using alignment charts such as this, it is found convenient to pin the chart on a drawing-board or similar plane surface. A fine black thread is used for the alignment, the ends of the thread being fastened to drawing-pins near P and Q. The length of the thread should be slightly greater than the height plus the diagonal of the diagram. This small detail of fixing the ends of the thread makes a very great difference in the convenience of using the diagram when many points are required.

My thanks are due to Professors Payne and Kernot of the Melbourne

University Engineering School for giving me facilities for plotting the curves, and to Miss L. M. Lloyd, who has assisted me in preparing the charts.

Melbourne Observatory :
1928 *November* 26.

A limited number of reproductions of the chart referred to in this paper has been made. Copies can be obtained by application to the Assistant Secretary, R.A.S.

Solar Streams of Corpuscles: Their Geometry, Absorption of Light, and Penetration. By S. Chapman, M.A., D.Sc., F.R.S.

1. In connection with magnetic storms and auroræ it is of interest to examine the motion of a stream of particles continuously emitted from a point on the sun. In the first instance the emitting point P will be supposed to lie in the sun's equator ; the removal of this restriction is considered in § 15.

Let a($= 7{\cdot}10^{10}$ cm.) denote the sun's radius, and R($= 1{\cdot}5 \,.\, 10^{13}$ cm.) the distance of the earth from the sun. Let Ω denote the angular velocity of the sun, taken to correspond with Carrington's sidereal rotation period of 25·4 days (or a mean synodic rotation period of 27·3 days). Then $\Omega a = 2{\cdot}10^5$ cm./sec. approximately. Let Ω' denote the mean angular velocity of the earth in its orbit round the sun.

Since

$$(1) \quad . \quad . \quad . \quad R/a = 216, \qquad (R/a)^{\frac{1}{2}} = 14{\cdot}7,$$

the reciprocals of these quantities can usually be ignored in comparison with unity.

Let r, θ denote polar co-ordinates in the sun's equatorial plane, referred to the sun's centre O and a fixed initial radius in this plane.

The point P of emission is supposed to be in heliographic longitude λ, referred to some standard solar meridian which at time $t = 0$ has the azimuth $\theta = \Theta_0$. Then for P at time t, the azimuth (Θ) is given by

$$(2) \quad . \quad . \quad . \quad . \quad \Theta = \Theta_0 + \lambda + \Omega t.$$

Ignoring the influence of the slight obliquity (7°·3) of the ecliptic to the sun's equator, the azimuth θ' of the earth at time t is given by

$$(3) \quad . \quad . \quad . \quad . \quad \theta' = \Omega' t + \theta_0',$$

where θ_0' is the azimuth at zero time.

The time t_c at which P is on the sun's central meridian as viewed from the earth, *i.e.* at which $\Theta = \theta'$, is given by

$$(4) \quad . \quad . \quad . \quad (\Omega - \Omega')t_c = \theta' - \Theta_0 - \lambda.$$

2. It will be supposed that the radial velocity v_0 of emission of the particles is constant, and likewise the transverse velocity, denoted by

$\omega_0 a$. After emission the force (if any) on the particles will be supposed radial, so that the angular momentum about O is constant, and the transverse velocity at distance r, denoted by ωr, is given by

$$(5) \quad . \quad . \quad . \quad . \quad \omega r^2 = \omega_0 a^2.$$

The radial velocity v at distance r will be some function $f(r)$ of r, depending on the nature of the radial force.

3. Consider the motion of the particle emitted at time t_0. Let r, θ denote its heliocentric co-ordinates at time t. At t_0 we have $r = a$, $\theta = \Theta$, $\dot{r} = v_0$, $\dot{\theta} = \omega_0$; at time t we have $\dot{r} = v = f(r)$, $\theta = \omega = \omega_0 a^2/r^2$. The geometrical equation of the path is found by eliminating the time from the last two equations, giving

$$d\theta/dr = \dot{\theta}/\dot{r} = \omega_0 a^2/r^2 f(r),$$

or

$$(6) \quad . \quad . \quad \theta = \Theta_0 + \lambda + \Omega t_0 + \omega_0 a^2 \int_a^r dr/r^2 f(r).$$

Thus θ is a function of λ, t_0, ω_0, and r, but the *form* of the path is independent of Θ_0, λ, t_0, which merely determine its orientation as a whole.

The inclination ϕ, at any point of the path, to the radius through the point, is given by

$$(7) \quad . \quad . \quad \tan\phi = r d\theta/dr = \omega_0 a^2/rf(r) = \omega_0 a^2/vr.$$

The time t of arrival at the distance r is given by

$$(8) \quad . \quad . \quad . \quad . \quad t = t_0 + \int_a^r dr/f(r),$$

the second term giving the duration of travel. From (8) r may be obtained as a function of t, and then (6) gives θ as a function of t.

Let Δt denote the time of passage to the earth's distance R; then

$$(9) \quad . \quad . \quad . \quad . \quad \Delta t = \int_a^{\mathrm{R}} dr/f(r).$$

The value ϕ' of ϕ at this distance is given by

$$(10) \quad . \quad . \quad . \quad . \quad \tan\phi' = \omega_0 a^2/v\mathrm{R}.$$

The angle $\Delta\theta$ subtended at O by the part of the path up to the distance R is given by

$$(11) \quad . \quad . \quad . \quad \Delta\theta = \omega_0 a^2 \int_a^{\mathrm{R}} dr/r^2 f(r).$$

3*a*. If the particles are subject to no force after emission, $f(r) = v_0$ always, and (6), (9), (11) take the approximate special forms

$$(6a) \quad . \quad . \quad . \quad \theta = \Theta_0 + \lambda + \Omega t_0 - \omega_0 a/v_0,$$

$$(9a) \quad . \quad . \quad . \quad \Delta t = \mathrm{R}/v_0,$$

$$(11a) \quad . \quad . \quad . \quad \Delta\theta = \omega_0 a/v_0 = \omega_0 a \Delta t/\mathrm{R},$$

where factors $1 - a/\mathrm{R}$ have been omitted (*cf.* § 1).

4. E. A. Milne * has shown that in the high levels where selective radiation forms the chief support of the solar atmosphere, the Doppler effect will enable upward-moving atoms to climb out of the absorption lines associated with them, and to be accelerated away from the sun. Beyond the distance $r = \frac{1}{5}a$ approximately, the variation of velocity is given by

$$(12) \quad . \quad . \quad . \quad . \quad v^2 = v_\infty^2\left(1 - \frac{8}{7}\frac{a}{r}\right),$$

where v_∞ is the final value of v at infinite distance. He finds that v_∞ is of the order $1{\cdot}6 \cdot 10^8$ cm./sec., roughly the same for all atoms, whatever the Fraunhofer line with which they are associated.

In this case the radial motion can readily be determined in terms of hyperbolic functions, but the formulæ will not be written down here. From them I calculate that $v = 0{\cdot}94v_\infty$ at $r = 10a$, so that over much the greater part of the journey from the sun to distance R the motion is nearly uniform. At $r = \frac{1}{5}a$, where the atom becomes clear of its absorption line, the velocity is about $\frac{1}{3}v_\infty$; this distance is attained in about 15 minutes. The distance R is attained in about 1·11 day, according to (12) and using Milne's value of v_∞. If the velocity were v_∞ at all distances, Δt would be about 1·09 day. In this and many other respects (though not all) the actual case, according to Milne's theory, can be illustrated very approximately by the case of uniform motion.

5. If the sun emits continuously from the point λ, the particles which have issued prior to any time t will at this instant lie on a certain curve. Each point on this curve will correspond to a particular earlier instant t_0 of emission, and t_0 will be a parameter in terms of which (and of t) the co-ordinates r, θ of the point can be expressed. The polar equation of the curve can be obtained on eliminating t_0 between r and θ, *e.g.*, by substituting in (6) the value of t_0 given by (8); thus

$$(13) \quad . \quad \theta = \Theta_0 + \lambda + \Omega t + \omega_0 a^2 \int_a^r dr/r^2 f(r) - \Omega \int_a^r dr/f(r)$$
$$\equiv \Theta_0 + \lambda + \Omega t + \mathrm{F}(r),$$

where $\mathrm{F}(r)$ is independent of λ and t.

Like the path of an individual particle, the *form* of the stream is independent of λ and t, which merely determine its orientation in the sun's equatorial plane. The stream-curve revolves with the angular velocity Ω of the emitting point.

The inclination χ of the stream, to the radius through the point r, θ is given by

$$\tan\chi = rd\theta/dr = rd\mathrm{F}(r)/dr$$
$$= -\{\Omega r/f(r)\} + \omega_0 a^2/rf(r).$$

At $r = \mathrm{R}$, writing $\chi \equiv \chi'$, we have

$$(14) \quad . \quad . \quad \tan\chi' = -(\Omega\mathrm{R}/v)(1 - \omega_0 a^2/\Omega\mathrm{R}^2).$$

* *M.N.*, **86**, 459, 578, 1926.

The angle subtended at the sun's centre by the part of the stream up to distance R will be denoted by $\Delta\Theta$; it is the longitude of the emitting point relative to the sun's central meridian, at the instant when $\theta = \theta'$. Clearly

$$(15) \quad \Delta\Theta = \mathrm{F(R)} - \mathrm{F}(a) = \omega_0 a^2 \int_a^{\mathrm{R}} dr/r^2 f(r) - \Omega \int_a^{\mathrm{R}} dr/f(r).$$

If t' is the time at which the stream, if undeflected by the earth's magnetic field, would pass through the earth's centre, *i.e.* when θ in (13) is equal to θ' in (4),

$$(\Omega - \Omega')t' = \theta_0' - \Theta_0 - \lambda - \mathrm{F(R)},$$

so that, by (5),

$$(16) \quad (\Omega - \Omega')(t' - t_c) = -\mathrm{F(R)}.$$

This gives the time between the passage of the emitting point across the sun's central meridian, and the instant of impact of the stream upon the earth.

5*a*. If the particles move with uniform velocity, as in § 3*a*, (13), (15), (16) have the following special forms :—

$$(13a) \quad \theta = \Theta_0 + \lambda + \Omega t - (\Omega r/v_0)(1 - a/r)(1 - \omega_0 a/\Omega r),$$

$$(15a) \quad \Delta\Theta = -(\Omega \mathrm{R}/v_0)(1 - \omega_0 a/\Omega \mathrm{R}),$$

$$(16a) \quad t' - t_c = \frac{\Omega}{\Omega - \Omega'}\left(1 - \frac{\omega_0 a}{\Omega \mathrm{R}}\right)\Delta t,$$

the last two being approximate.

Similar formulæ can be worked out, in terms of hyperbolic functions, for streams emitted in accordance with Milne's theory (§ 4); the formulæ give results agreeing fairly closely with (13–16*a*).

6. The following tables give some numerical illustrations of the above results.

The peripheral velocity Ωa of the (equatorial) point of emission is 2·0 km./sec. The tangential velocity $\omega_0 a$ of the emitted particles will probably not exceed 100 km./sec., so that ω_0/Ω is likely to lie between $\pm$ 50. Hence in (14) the second factor may be omitted, and

$$\tan \chi' = -\Omega \mathrm{R}/v = -4{\cdot}3 \cdot 10^7/v,$$

reckoning v in cm./sec.; (10) may then be written as

$$\tan \phi' = -(\omega_0/\Omega)(a^2/\mathrm{R}^2) \tan \chi',$$

showing that ϕ' is always very small compared with χ'. Since χ' is always negative, the stream at distance R lags behind the rotating radius through the point of emission; ϕ' has the same sign as ω_0, and if ω_0 is positive the paths of the particles are inclined slightly forward.

Values of χ' corresponding to a wide range of values of v are given

in Table I. They are practically independent of ω_0, with the assigned limits for the latter. The corresponding values of ϕ' are proportional to ω_0, but even when $\omega_0 = \pm 50\Omega$ they never exceed 0°·3.

Table I.

v (cm./sec.)	10^7	5.10^7	10^8	$1{\cdot}6.10^8$	5.10^8	10^9	10^{10}
$-\chi'$	77°	41°	23°	15°	5°	2°·3	0°·2

The values of $\Delta\theta$ and $\Delta\Theta$ depend appreciably on ω_0. In Table II. results are given for the case $\omega_0 = \Omega$, and for various uniform velocities of radial motion.

Table II.

v (cm./sec.)	10^7	5.10^7	10^8	$1{\cdot}6.10^8$	5.10^8	10^9	10^{10}
$\Delta\theta$ (11*a*)	1°·1	0°·6	0°·1	0°·1	0°·1	0°·0	0°·0
$-\Delta\Theta$ (15*a*)	245°	122°	24°	15°	12°	2°·4	0°·2
Δt (days) (9*a*)	17·4	8·7	1·7	1·1	0·9	0·17	0·017
$t'-t_c$ (days) (16*a*)	18·7	9·3	1·9	1·2	1·0	0·19	0·019

In the case of particles accelerated according to Milne's theory, the values in the above table will be only slightly altered, if v is taken equal to v_∞ ; this has already been mentioned in § 4 as regards Δt, while a calculation of $-\Delta\Theta$ on the basis of Milne's formulæ for v as a function of r, taking $v_\infty = 1{\cdot}6 \cdot 10^8$, gives 15°·1, agreeing closely with the corresponding value in Table II.

The times of passage (Δt) of the particles up to the earth's orbit, or the values of $t' - t_c$ for the interval between the transit of the emitting centre across the sun's central meridian, and the time of arrival of the particles at distance R, for the value $v = 1{\cdot}6 \cdot 10^8$, agree in order of magnitude with the mean interval—about $2\frac{1}{2}$ days—found by Maurain * between days of notable terrestrial magnetic disturbance and the epoch of maximum sunspot number in the neighbouring period. Abetti † and Greaves and Newton ‡ have found similar intervals, of 1·1 and 1·5 days respectively, between notable solar activity, or the central passage of large sunspots, and the commencement of great magnetic storms.

7. If such a stream of corpuscles as has been considered were undeflected by the earth's magnetic field, it would first impinge upon the earth's equator in a meridian upon the sunlit hemisphere, differing in longitude from the twilight meridian by the angle $-\chi'$; the obliquity of the axes of the earth and sun to the ecliptic is here neglected, but the necessary modification when this is allowed for can easily be made. For $v = 1{\cdot}6 \cdot 10^8$, $-\chi' = 15°$, corresponding to the meridian 17^h, reckoned from midnight; for $v = 10^8$, $5 \cdot 10^8$, the corresponding local times are $16\frac{1}{2}^h$ and $17\frac{2}{3}^h$.

* *Ann. de l'Inst. Phys. du Globe, Paris*, **5**, 87, 1927.
† *Second Report on Solar and Terrestrial Relationships*, p. 9, 1929.
‡ *M.N.*, **88**, 556 ; **89**, 84, 1928.

8. Some years ago I discussed the possible differences between the times * of sudden commencement of magnetic disturbance at different terrestrial stations, on the assumption that the storms are produced by streams of particles such as are here considered. Without adopting any particular theory as to the detailed mechanism of magnetic storms, it was pointed out, apparently for the first time, that the relative times of commencement of a storm, at different stations, should depend mainly on the orientation of the earth, at the instant, relative to the sun, and that therefore the *local times* of commencement should form an important part of the basis of discussion. It was also stated that the stream would overtake the earth on the P.M. (post meridiem) side; the present calculations show that if Δt is of the order of 1 day, the stream, if undeflected by the earth's magnetic field, would first meet the earth on a sunlit meridian about an hour in advance of sunset.

9. In a recent paper L. Rodés † has maintained that the earth will overtake the clouds or streams of particles from the sun, so that magnetic storms should commence first near the sunrise meridian. He suggests that my view implies a transverse velocity of the particles, at distance R, greater than the orbital velocity of the earth, and 200 times as large as their initial transverse velocity, the principle of conservation of angular momentum therefore being grossly violated. In view of this misconception, which has also appeared in the writings of other authors, it seems not superfluous to show explicitly, as in §§ 3, 5, that though the motions of the individual particles are almost radial at great distances, the form of the stream composed of all the particles successively emitted rotates with the sun.

The 27·3 day recurrence-tendency in magnetic conditions, established by the work of Maunder ‡ and Chree,§ suggests that magnetic disturbance is due to *long-continuing* solar emissions, which, however, affect the earth only when they come near it; they must in general overtake it in its orbit, in the manner described above. The earth and the stream of solar particles could approach each other in the opposite direction, so as to meet first near the sunrise meridian, only if θ in (13), at $r = R$, increases more slowly than $\Omega' t$; that is, if $\Omega' t - \theta$ is positive and increasing. This requires that $\Theta_0 + \lambda + F(R)$ is variable, and decreasing faster than $(\Omega - \Omega')t$; for this to occur, either λ or ω_0, or both, must be decreasing. If ω_0 is constant, λ must be decreasing faster than $(\Omega - \Omega')t$; that is, the emitting centre must actually be moving backwards on the sun relative to the earth; such large velocities of the disturbed regions must be rare. The other alternative, that ω_0 is decreasing (algebraically), could obtain only for a short interval, since ω_0 is limited; this brief interval, moreover, would have to coincide with the interval of emission of the part of the stream that crosses the earth's orbit near the earth. Such coincidences must be rare, and, even so, the stream would already previously have swept forward across

* *Proc. Phys. Soc.*, **30**, 205, 1918.
† *Terr. Mag.*, **32**, 126, 1927.
‡ *M.N.*, **65**, 2, 538, 666, 1904.
§ *Phil. Trans.*, A, **212**, 75, 1912; **213**, 245, 1913.

the earth, unless this particular part was the head of the stream, involving another coincidence. So long as λ and ω_0 are constant, θ at $r = R$ increases at the rate Ωt.

10. Owing to the magnetic deflection of the stream near the earth, the stream may not first impinge on the meridian indicated in § 7, and the commencement of the magnetic storm may not travel across the earth at the same rate as the stream. It seems possible that the storm may commence a few hours before the body of the stream reaches the earth.

11. In Table II. the values of $\Delta\Theta$ and $t' - t_c$ depend materially on the assumption that $\omega_0 = \Omega$, *i.e.* that the emission is radial relative to the sun. If, however, ω_0/Ω has the suggested extreme values + 50 or − 50, the corresponding values of $\Delta\Theta$, for $v = 1{\cdot}6 \cdot 10^8$, are 12° and 18°, while the values of $t' - t_c$ are 0·9 and 1·5 days; $\Delta\theta$, referring to the paths of the particles, is multiplied by ω_0/Ω, and therefore for $v = 1{\cdot}6 \cdot 10^8$ would be $\pm 3°{\cdot}7$. These results are for the case of a uniform velocity. In accelerated emission according to Milne's theory, the values of $\Delta\theta$ and $\Delta\Theta$ would be more affected, but most of the alteration in the change in azimuth would occur near the sun. Moreover, S. R. Pike * has shown that if the particles have a horizontal velocity relative to the sun they will be subjected to a horizontal acceleration, even in the absence of local inequalities of temperature on the sun's surface. But the horizontal acceleration will decrease outwards even more rapidly than the vertical acceleration, particularly since, however oblique to the vertical may be the original motion, the further it travels away from the sun, the smaller the ratio borne by the horizontal to the vertical component of velocity. Here we are concerned only with particles which leave the sun altogether, whereas Pike's calculations mainly dealt with clouds in the sun's atmosphere that rise and fall back again. It is desirable to extend his work to the motion of particles in the interval $r = \frac{1}{10}a$ to $r = 10a$.

Unless the solar emissions occur in a direction systematically different from the vertical, observed values of $t' - t_c$ may be used to infer v from Table II., though irregularities in the direction of emission may cause a considerable spread of the values of $t' - t_c$ for a single value of v.

12. The lateral thermal motions of the emitted particles will cause a spread in the values of ω_0 and $t' - t_c$, which will vary with the mass of the atoms. For hydrogen and calcium, at 5000°, the thermal velocities are respectively 11 and 1·8 km./sec., which are small compared with v_∞; the corresponding values of ω_0/Ω are $\pm$ 5·5 and $\pm$ 0·9. The effect on $t' - t_c$ is of the order 0·1 day or less, and on $\Delta\Theta$ of the order 1° or less; these quantities are too small to require further consideration at present.

13. Up to this stage the discussion has referred mainly to a one-dimensional or linear stream of particles emitted under constant conditions from a particular point on the sun. If particles are similarly emitted over a range $\delta\lambda$ of longitude on the sun, there will be a corresponding range in θ for the stream at all radii r, as given by (13). Hence

* *M.N.*, **88**, 3, 1927.

the breadth of the stream, $r\delta\theta$ or $r\delta\lambda$, will be proportional to r, and there will be a lateral expansion in the ratio R/a, or 216, in the passage from the sun to the earth's distance R. There will be a similar expansion perpendicular to the equatorial plane, corresponding to a range δl in latitude of the emitting area. Hence the original density of the stream will be reduced, on this account, in the ratio $(a/R)^2$, or about 1/46,000, on reaching the earth's orbit.

Expansion due to this cause will be in nearly the same ratio whatever the velocity, and whatever the values of δl and $\delta\lambda$, so long as these do not exceed about 20°. There will also be an expansion due to the thermal motions of the atoms of the stream. For hydrogen and calcium atoms respectively, with thermal velocities of 11 and 1·8 km./sec., the total broadening of the stream during the 1·1 day occupied in the journey to the earth (if $v = 1{\cdot}6 \cdot 10^8$ cm./sec.) will be $2{\cdot}1 \cdot 10^{11}$ and $3{\cdot}4 \cdot 10^{10}$ cm., and more or less in inverse proportion to the velocity v, if this is not $1{\cdot}6 \cdot 10^8$. This broadening, which will be superposed on the expansion previously considered, will represent a greater or less ratio of expansion according to the size of the stream. If $\delta\lambda$ and δl are equal to 1°, 5°, 10°, 15° respectively, the breadth of the stream, apart from the thermal expansion, at $r = R$, will be $3{\cdot}8 \cdot 10^{11}$, $1{\cdot}9 \cdot 10^{12}$, $3{\cdot}8 \cdot 10^{12}$, $5{\cdot}7 \cdot 10^{12}$ cm.; in the case of hydrogen atoms the above thermal broadening bears to this breadth the ratios 0·55, 0·11, 0·055, 0·037, while in the case of calcium atoms these ratios must be reduced by the factor 1·8/11 or 0·16. Clearly the thermal expansion is of minor importance except for streams of small angle, less than 5°.

In addition to these two causes of expansion, there is the initial divergence of the emission from any particular point in the emitting area, corresponding to a range in the value of $\omega_0 a$, the horizontal velocity (apart from the range, already considered, due to the thermal motions). It is difficult to say what this may be, but it will at any rate render the breadth of the stream, at $r = R$, greater than the estimates already made, which are therefore to be regarded as lower limits.

According to Pike (*loc. cit.*) the horizontal acceleration which, as he showed, must result if the particles have any horizontal motion at emission, would rapidly disperse upward-moving clouds in the sun's atmosphere; but this conclusion scarcely seems to be justified. On the whole the emitted particles will have the same mean motion, and their horizontal (and vertical) accelerations will therefore be nearly uniform, and the cloud will move as a whole; examination shows that, at least if the only relative motions of the particles are their thermal velocities, the spread of the corresponding horizontal accelerations will not seriously disperse a cloud which is able to escape altogether from the sun. Photographs of escaping prominences seem to support this conclusion.

It is clear, however, that the solar streams are likely to be of great extent where they cross the earth's orbit; the minimum breadth due to thermal expansion alone, in the relatively unfavourable case of a calcium stream, is $3{\cdot}4 \cdot 10^{10}$ cm., or 50 earth-radii. A stream of this breadth would take about 25 minutes to sweep across the earth. Magnetic storms last much longer than this, sometimes continuing for

three days or more. If this indicates the order of magnitude of the time during which the earth is enveloped by the stream (which is likely, though not certain), the breadth of the stream at distance R from the sun must sometimes be so great as 10^{13} cm., subtending an angle of about 40° at the sun. This great breadth can occur only if either the emitting area on the sun has a comparable angular breadth, or if the angle of divergence of the stream on emission is of this order, or if both causes operate together. Thermal expansion is likely to be a very minor factor in any stream which envelops the earth for several hours.

14. Milne's theory of corpuscular emission from the sun indicates that there must be a continual escape of particles from all over the sun, but without some additional hypothesis, such as the assumption of a local excess of temperature in the reversing layer or photosphere, it does not account for the emission of a definite laterally limited stream, which seems necessary to explain the 27-day recurrence-tendency in terrestrial magnetic conditions. The emission, and consequently the local excess of temperature, would appear to be of long continuance. At present it is not clear how such a continued excess could be maintained.

The identification of any definite observed bright markings on the sun with corpuscular emissions towards the earth, or their association with particular geophysical phenomena, has not yet been achieved. In most of the past attempts to correlate terrestrial magnetic with solar phenomena, sunspot data have been used. It is clear, from the observed occurrence of magnetic disturbance at times when no sunspots have been visible for days, that sunspots are not an essential accompaniment of magnetic disturbance, but there is undoubtedly on the whole a considerable degree of association between magnetic disturbance and the presence of sunspots near the centre of the sun's disc. This is rendered somewhat strange by Pike's demonstration that dark areas on the sun, such as spots, have the effect of drawing in or "attracting" chromospheric matter or prominences. They must therefore reduce the chance of escape of corpuscles propelled upwards by neighbouring bright areas. It would seem that bright areas would be most effective as emitters when unaccompanied by sunspots, and that the observed connection of magnetic disturbance with sunspots is due to the infrequency of bright areas except in association with sunspots.

15. If the situation of the bright emitting areas is similar to that of sunspots, these areas must occur most often in latitudes 10° to 15°, though all latitudes up to ± 30°, or even beyond, will be affected. Unless emission occurs in a direction parallel to the sun's equatorial plane, the stream will not lie in a plane, but will describe a surface of revolution about the sun's axis. Up to the distance R, for $v > 10^8$, this surface will be nearly conical, since the stream is nearly straight. The rate of revolution of the stream round the sun's axis is Ω, whatever the point or direction of emission, so long as the conditions of emission are constant. The position and extent of the stream at large distances (such as R) from the sun depend mainly on the *direction* of emission, and only to a minor degree on the precise situation of the emitting point. Until there is observational evidence to the contrary, it is natural to

assume that on the average the emission from any point is radial, though there may well be considerable deviations in particular cases.

Consider a conical stream of semi-angle α, emitted so that its axis is inclined at the angle l to the sun's equator; if the emission is radial, the emitting point must be in heliographic latitude l, but this is not necessary. The stream will sweep over any distant object of which the heliographic latitude lies between $l \pm \alpha$; the latitude (l') of the earth must probably lie between these limits, if a magnetic storm is to occur. The ecliptic is inclined at 7°·3 to the sun's equator, and the earth crosses the sun's equator on about June 5 and December 5, *i.e.* near the solstices, and attains maximum northerly and southerly heliographic latitudes 7°·3 on about September 5 and March 5. Near the former dates, when $l' = 0$, streams radially emitted from areas in latitude $\pm 12\frac{1}{2}°$ must be of angular breadth not less than 25° in order to envelop the earth. When $l' = 7°{\cdot}3$ N., the angular breadth 2α need only be 11° for streams radially emitted from latitude $12\frac{1}{2}°$ N., but for streams from an equal southerly latitude it must be at least 40°. It would therefore seem likely that in September magnetic disturbance is commonly due to northern solar emitting areas, and in March to southern areas. It would be of interest to examine whether the magnetic and solar data accord with this conclusion.

It is well known that the frequency and intensity of terrestrial magnetic disturbance have two annual maxima, near the equinoxes, with intervening minima near the solstices. These maxima and minima are simultaneous in the earth's northern and southern hemispheres, showing that they are not due, like the ordinary terrestrial seasons, to the annual change in the *orientation* of the *earth* relative to the sun. The annual variation of magnetic disturbance relates to the earth as a whole, and recurs year by year as the earth revolves round the sun. Hence it may reasonably be attributed to the changing position of the earth as a whole, relative to the sun: or, what is equivalent, to the changing orientation of the *sun* relative to the earth. It depends not on the sun's geocentric co-ordinates, but on the earth's heliocentric co-ordinates, and particularly on its heliographic latitude (the earth's heliographic longitude is affected more by the sun's rotation than by the earth's orbital motion, and the changes in longitude are therefore associated with the 27-day recurrence-tendency in terrestrial magnetic conditions). This view accords with the suggestion made by Cortie * that the maxima of magnetic disturbance occur at the equinoxes because then the radius from the sun to the earth is nearest to one or other maximum sunspot zone.

If on the average the emissions are radial, it is necessary, if this explanation is to be satisfactory, that streams of which the angular breadth exceeds 11° shall be more than twice as numerous as those of which the angular breadth exceeds 25°; for at the equinoxes all the former, from one zone only (if we may ignore the few, of breadth greater than 40°, from the other zone), will sweep across the earth,

* *M.N.*, **73**, 52, 1912.

while near the solstices streams from both zones equally will traverse the earth, but only if their angular breadth exceeds 25° : and the conjunctions are more frequent near the equinoxes than near the solstices. The restriction is only crudely stated in this form, because of the spread in the latitudes of the emitting areas, and in the directions of emission relative to the radius through the point ; it is scarcely worth while at present to formulate the statement more exactly. A restriction of the kind indicated does not seem at all improbable *a priori*.

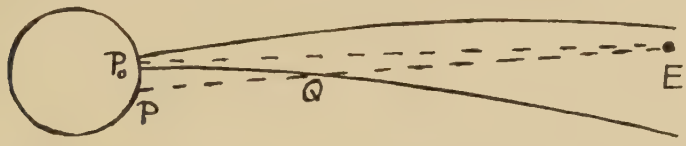
16. The density of the streams of solar corpuscles associated with terrestrial magnetic disturbance and auroræ is not yet known by inference either from the solar conditions or from geophysical considerations. In the sun's reversing layer the number of atoms per c.c. ranges from about 10^{14} to 10^{6}, and the density of the solar streams at emission perhaps lies within these wide limits ; the expansion during the passage to the distance R is at least in the ratio 46,000, so that the number of atoms per c.c. of the stream at this distance will have the corresponding limits $2 \cdot 10^{9}$ and 20.

It seems of interest to point out that there is a possibility of determining the density directly, by the study and comparison of the solar spectrum at times of magnetic quiet and of magnetic disturbance. During the latter periods the earth is probably enveloped in the stream for many hours, sometimes even for days, and the whole of the sun's disc must then be viewed through a great length of the stream, different, however, for different parts of the disc. By hypothesis the atoms of the stream are of the same kind as those that produce the Fraunhofer lines, and after leaving the sun they will continue to absorb, but in light of wave-length modified by the Doppler displacement corresponding to their velocity, which will vary along the stream. The extreme velocity, $1 \cdot 6 \cdot 10^{8}$ cm./sec. (according to Milne's calculation), corresponds to a shift of the K line of Ca^{+} by about 22A, and the shifts for other important lines will be of the same order of magnitude. Since it appears that from $r = 10a$ the atoms have at least 94 per cent. of their final velocity, the stream will absorb approximately in the same displaced wave-length (for each species of atom, associated with a particular Fraunhofer line) throughout all but the small initial fraction of its length up to $r = 10a$.

The Fraunhofer lines and the adjacent continuous spectrum on the violet side will have a certain definite contour, different for different parts of the sun's surface, according as it is normal or disturbed. It should be possible to ascertain the normal undisturbed form (which may depend on the distance of the observed point from the centre of the disc) by spectrograms taken at quiet solar periods ; comparison of the spectrum for any region and time with this normal form would afford an indication whether this region was disturbed, but this alone might not indicate whether the disturbance was of the kind associated with the expulsion from the sun of streams of corpuscles capable of affecting the earth, or whether it was associated only with local phenomena of the sun's atmosphere.

Now suppose the spectrum of various parts of the sun's disc is

examined at a time when, from the existence of a magnetic storm, the earth E is believed to be enveloped within a solar stream, the lateral dimensions of which are of the order 10^{12} or 10^{13}. Let P_0 be the point of emission of the stream on the sun. The line of sight EP_0 may be expected to lie wholly within the stream, if E is well inside, and if the



stream is not too curved (that is, if v is not too small). The spectrum of the light proceeding from P_0 and viewed at E will differ from that of the same light as it would appear before the stream (a few hours previously) enveloped the earth, owing to the additional absorption by the atoms along the whole length of the stream. The effect would be to extend the absorption lines on the violet side, to a distance from the centre of, for example, the K line, of 22A; the mode of extension of the line will depend on the law of variation of velocity and density along the stream. As the final velocity is closely approached already at $r = 10a$, the density of the stream beyond this distance will decrease approximately as $1/r^2$, owing to the angular divergence of the stream; for $r < 10a$, the variation will be still more rapid, owing to the relatively rapid change of velocity along the stream; the conservation of matter requires that as the velocity of the atoms increases, the density decreases in inverse ratio, apart from any other causes of decrease. If the density varied as $1/r^2$ along the whole length of the stream, the number of atoms contained in a column of 1 sq. cm. section along EP_0 would be 9 times as great between $r = a$ and $r = 10a$ as between $r = 10a$ and $r = R$; for the reason first stated, the actual ratio will exceed 9. The corresponding ratio for the sections between $r = \frac{4}{3}a$ (where the atoms get clear of the normal absorption line), $r = 10a$, and $r = R$ is 6·5, which is an underestimate. This implies that in the range of absorption extending 14A (allowing for the half-width 8A of the line) beyond the K line, about $\frac{1}{7}$ to $\frac{1}{10}$ of the absorption occurs in the last 1A or $1\frac{1}{2}$A corresponding to the range of v from $0{\cdot}94v_\infty$ to v_∞, or of r from $10a$ to R. Since the atomic absorption coefficient is approximately known, a measurement of the intensity of this extension of the absorption line should enable the number of the corresponding atoms contained in the stream to be inferred.

This measurement might be rendered uncertain, however, because of the disturbed nature of the spectrum for the region P_0. A more reliable method therefore seems to be to examine the spectrum of some undisturbed region P of the sun, while the earth E is within the stream. The line of sight EP will not be wholly in the stream: at some point Q it will cross the boundary of the stream, and the part PQ will be outside the stream. The absorption will therefore be due to the atoms between E and Q, which will have velocities ranging from

v_Q (say) to v_∞. If Q is beyond $\frac{4}{3}a$, v_Q will correspond to a Doppler displacement of more than the half-width of the normal absorption line, and the spectrum of the light from P will show a new absorption line on the violet side of the normal line, and separated from it by a greater or less interval, the greater the distance PQ. The closer P is to P_0, the smaller PQ will be, and the wider the absorption line added to the normal spectrum for the region P; in general P_0 may be expected to lie near the centre of the sun's disc (usually some degrees past the central meridian). For different points P, the additional absorption should be the same in intensity over the common parts of the line, extending from the violet end, though the width of the new line, or its separation from the normal line, will depend on the position of P.

If n' is the number of atoms of the species associated with the K line (for example), at the distance $r = R$, the number traversed by the line of sight EP is approximately $Rn'(R/r - 1)$ per sq. cm. cross-section of the beam, if r refers to the point Q. If $r = 10a$, this is approximately $20Rn'$, or $3 \cdot 10^{14} n'$. Eddington,* discussing absorption by diffuse matter in stellar space, on the basis of Milne's theory of chromospheric absorption, estimated that a moderately dark H or K line would be produced by a column containing $1 \cdot 5 \cdot 10^{13}$ Ca^+ atoms per sq. cm. cross-section. Hence if the solar stream were composed wholly of Ca^+ atoms (and the corresponding free electrons), and if their number per c.c. near the earth were only 1/20, a measurable absorption line, 1 or 2A wide, would be produced in the spectrum of the light from such a point P. If P were nearer P_0, so that $PQ < 10a$, more atoms would contribute to the absorption, and the line would be wider. It seems not improbable that n' exceeds 1/20, so that there would appear to be hope of detecting the absorption due to the stream, provided the sun's spectrum from normal regions is sufficiently steady, and that the absorption is not obscured by neighbouring normal lines; at least the matter seems deserving of observational test.

17. It is of interest to examine the extreme interval between the normal absorption line and the line due to the stream, in a typical case when the earth is within the stream. Suppose that at the instant considered the "following" or western edge of the emitting area is on the sun's central meridian. For this to occur, when the earth is within the stream, the latter must be at least 15° wide (assuming $v_\infty = 1 \cdot 6 \cdot 10^8$). In the equation (13) for the following edge of the stream, the azimuth Θ, or $\Theta_0 + \lambda + \Omega t$, of the emitting point, is equal to θ', the azimuth of the earth. Hence the angle subtended at the sun's centre by the part of the stream up to the distance r is $\theta' - \theta$, and the perpendicular distance of the point r, θ from the radius through the earth is $r(\theta' - \theta)$. Let r refer to the point in which this western boundary of the stream is cut by the line of sight from the earth to the equatorial point on the sun's western limb. Then by similar triangles

$$r(\theta' - \theta)/a = (R - r)/R.$$

* *The Internal Constitution of the Stars,* p. 381, 1926.

For a stream of uniform velocity $1{\cdot}6 \cdot 10^8$ cm., radially emitted ($\omega_0 = \Omega$), (13*a*) leads to the solution $r = 27a$ approximately; the line of sight to the western limb would in this case traverse only about $7Rn'$ atoms per sq. cm. The result for a stream in which the velocity varies according to Milne's theory is very similar.

If the line of sight is to a point P midway between the centre and the western limb, $r = 20a$ approximately. The position of P corresponding to $r = 10a$ is about $\frac{1}{10}$ the radius of the disc from the centre. In all these cases the additional absorption line would be narrow and widely separated (by nearly 14A) from the normal absorption line.

18. The observation of the additional absorption line would also check the theoretical value of the velocity of the stream, which Milne estimates to be about $1{\cdot}6 \cdot 10^8$ cm./sec. This (§ 6) is in fair accord with the interval between the passage of markedly active solar regions across the central meridian, and the advent of supposedly associated magnetic disturbance on the earth. There is, however, a difficulty, to which he drew attention, in explaining how particles with no greater speed than this can penetrate so far down into the earth's atmosphere as the observed lower limit of auroral light, which is sometimes as low as 80 km. After reviewing the experimental data bearing on the penetration of air by calcium atoms with this velocity, he concluded that their equivalent air-range (that is, the range in air of the density existing at ground-level in the earth's atmosphere) is about 0·15 cm. He pointed out that according to the upper-atmospheric calculations by Chapman and Milne,* the equivalent air-ranges down to 100 km. and 80 km. height respectively, along a vertical path, are about 0·4 cm. and 5·6 cm., if there is no hydrogen in the upper air, or 1·2 cm. greater if hydrogen is present to the extent corresponding with the small traces near the ground. Even the least of these ranges, 0·4 cm., is thrice too large to be consistent with the supposed velocity of the calcium atoms. Actually, however, the difficulty is still greater than he remarked, because there seems little doubt that the densities calculated by Chapman and Milne are too small for heights above 80 km. The exact values are still uncertain, but the equivalent air-ranges down to 100 and 80 km. respectively should probably be increased by factors of about 4 and 2, to about 1·5 cm. and 10 cm., even if the possible (though perhaps now improbable) existence of helium, as well as hydrogen, in the upper atmosphere, is ignored. The latter range exceeds that of the fastest α-particles (7 cm.), while even the former is ten times as large as the range for the calcium atoms, as estimated by Milne.

This difficulty was discussed also by Lindemann † in 1919, in connection with his theory of solar clouds of ionized hydrogen, reaching the earth with a velocity $8{\cdot}10^7$ cm./sec. The estimated equivalent air-range was only 0·00037 cm., so that the difficulty of explaining penetration down to auroral levels in the earth's atmosphere was far greater than on Milne's theory. Lindemann at that time endeavoured to over-

* *Q.J. Roy. Met. Soc.*, **46**, 384, 1920.
† *Phil. Mag.*, **38**, 669, 1919.

come the difficulty firstly by supposing that there is very little hydrogen in the upper atmosphere, and also by supposing that the stratospheric temperature of about 220° is not maintained up to auroral levels, but that there the temperature is lower, and the density less. However, it is to Lindemann himself, with Dobson, that the important discovery, made shortly afterwards,* was due, that the temperature in the stratosphere increases upwards instead of decreasing, and for this reason the high-level densities calculated by Chapman and Milne, on the basis of a uniform stratospheric temperature of 220°, need to be increased.

The explanation of the height of auroræ consequently remains difficult, unless the velocity of the solar corpuscles is at least ten times as great as Milne's estimate, which seems unlikely. Only two other solutions, neither of which is attractive, seem possible. One is that the lower limits of auroral luminosity do not mark the depth of penetration of solar corpuscles; this would imply that the low auroræ are due to electric discharges of some kind, induced by corpuscular precipitation at higher levels; if so, it would seem difficult to account for the rather definite form of the auroral rays. The other possibility is that the extrapolation of experimental results on corpuscular penetration in air of ordinary or moderately low densities, to air of the extremely low densities existing above 80 km. height in the earth's atmosphere, is at fault, and that atoms can penetrate further than the law of equivalent masses would indicate, when the density is very low.

Summary.

The paper deals mainly with the geometry of streams of corpuscles continuously emitted from the sun, with special reference to the process of emission recognised by Professor Milne. It is pointed out that there is a possibility of determining the density and velocity of such a stream by observation of the line-absorption which it will produce in the spectrum of light from undisturbed regions of the sun, when these are viewed from the earth while the latter is enveloped in a stream, as is supposed to be the case during magnetic storms. The difficulty in accounting for the penetration of the corpuscles into the earth's atmosphere to the lowest observed auroral levels is discussed.

The Electrical State of Solar Streams of Corpuscles.

By S. Chapman and V. C. A. Ferraro.

1. The purpose of this paper is to discuss the electrical state of a stream of solar corpuscles when it reaches the earth's orbit, assuming that the process of emission is of the kind discovered by E. A. Milne.† The stream is supposed not to pass near the earth, so that the influence

* *Proc. R.S.*, A, **102**, 411, 1922.
† *M.N.*, **86**, 459, 578, 1926.

of the earth's magnetic field upon its motion does not arise ; this very difficult problem will be discussed in a later paper.

The problems here considered are in part similar to some of those discussed by F. A. Lindemann * in a " Note on the Theory of Magnetic Storms." He supposed a cloud of neutral hydrogen atoms to become accelerated away from the sun, with a final velocity of about $8 \cdot 10^7$ cm./sec., by radiation-pressure, and gradually to become ionized. Among other problems he considered the recombination, the expansion due to the thermal motions of the ions and electrons, and the small positive charge produced by the escape of electrons.

In Milne's theory the atoms when first projected may be neutral or ionized, consisting of all the kinds which show evidence of being supported by radiation-pressure in the chromosphere ; among the ionized atoms will be Ca^+, Ti^+, Sr^+, while the neutral ones will include *H*, *He*, *Na*, *Fe*, *Mg*. Some of the latter, particularly *H*, will become ionized before reaching the earth's orbit, but there may be an appreciable number which remain neutral throughout. These will travel in straight lines even in the earth's magnetic field ; except inasmuch as they slightly reduce the mean free paths of ions and electrons in the cloud they do not now concern us.

The cloud will tend to have a resultant positive charge because the non-neutral atoms emitted are all positive, while the acceleration of the electrons by radiation-pressure is relatively small. The positive charges will draw electrons after them from the sun's atmosphere by electrostatic attraction, but Milne suggested that this attraction may be only partly effective, so that the cloud as a whole may have a positive charge, or its head may be positive and its tail negative. In § 2 it is shown that the latter does occur, but only to a negligible degree even in the near neighbourhood of the sun. Further, it will be shown that in any case (so long as the velocity of the cloud is 10^9 cm./sec. or less) any net charge on the cloud will be dispersed by the time the earth's orbit is reached. It therefore seems that corpuscular theories of magnetic storms and auroræ must be based on streams or clouds that are almost exactly neutral as they approach the earth—unless, with Birkeland, it is supposed that the sun emits streams of corpuscles having velocities of 10^{10} cm./sec. or more ; but apart from radioactive processes, which are unlikely to be operative to a significant extent in the sun's atmosphere, there is no known means by which such high-speed particles could be emitted.

2. Professor Milne's suggestion that the head of the cloud may be positive and the tail negative was based upon the idea that the ions are subject to a much greater force ($m_i f_i$) of radiation-pressure than the electrons ($m_e f_e$) ; hence the latter will lag behind until a compensating electrostatic force is set up which neutralizes the difference of radiation-pressure, so that both move with the same acceleration f. It is readily shown that a minute displacement suffices for the purpose.

The cloud will be taken as spherical. The greatest relative separa-

* *Phil. Mag.*, **38**, 669, 1919.

tion of the charges will be obtained if we ignore altogether the radiation-pressure on the electrons. The separation of the charges is equivalent to a polarization in the direction of motion (x) of the clouds; the effect is to produce a surface distribution of charge (with no resultant volume charge) of density $\sigma \cos\theta$, at a point the radius to which, from the centre of the cloud, makes an angle θ with the direction x. The electrostatic field within the cloud is uniform, in the direction x, and of intensity $\frac{4}{3}\pi\sigma$. Hence the equations of motion of an ion and electron respectively are

$$m_i f_i - \tfrac{4}{3}\pi\sigma e = m_i f,$$
$$\tfrac{4}{3}\pi\sigma e = m_e f.$$

Consequently

$$\sigma = \frac{3}{4\pi e}\frac{m_i m_e}{m_i + m_e} f_i \doteqdot \frac{3m_e}{4\pi e} f_i.$$

An upper limit for σ will be obtained by assigning to f_i its maximum value, which occurs when the ion has just got clear of its absorption line and is still near the sun's surface; in Milne's formula * for $\ddot{r}$ (or f_i), *i.e.*,

$$f_i = \frac{I_1 - I_0}{I_0}\frac{a^2 g}{V}\frac{\dot{r}}{r^2},$$

we write $r = a$, $\dot{r} = V$, which give an overestimate of f_i. Taking $I_1/I_0 = 9$ (according to Milne's estimate), we have

$$f_i < 8g,$$

where $g = 2{\cdot}7 \cdot 10^4$ cm./sec.$^{-2}$. Hence

$$\sigma < \frac{6m_e g}{\pi e} = 10^{-13}.$$

This limit for σ corresponds to $2 \cdot 10^{-4}$ unit charges (positive or negative) per sq. cm. of surface of the cloud at its head and tail, and for a cloud of density N ions per c.c. would be produced by a relative separation of the ions and electrons by $2 \cdot 10^{-4}/N$ cm.; *e.g.* if $N = 10^9$, it is $2 \cdot 10^{-13}$ cm. Only if N were 10^{-5} or less, and the cloud were very small, could the two sets of charges be separated widely by radiation-pressure. It is clear that the separation of the charges by relative radiation-pressure is quite insignificant, and hence in the further work it is sufficient to consider only the resultant charge on the cloud.

3. The electrical state of a cloud or stream on reaching the earth's orbit can be inferred with sufficient exactness by considering a spherical cloud symmetrical about its centre, and of the same radius as the cross-section of the stream or cloud. The radius r_0 of the sphere at emission may be taken as between 10^8 cm. (1000 km.) and 10^{10} cm. (or $\frac{1}{7}$ the radius of the sun). For simplicity the presence of neutral atoms will be ignored, as they seem unlikely to outnumber the ions, and, if so, they will scarcely affect the subsequent calculations. The number of

* *Loc. cit.*, p. 464.

ions per c.c. at emission is unlikely to exceed 10^9. As the cloud approaches the earth's orbit it is likely to expand to at least 10^4 times its original radius (with corresponding decrease of density) owing to the thermal motions of the particles, and the probable initial divergence of the directions of projection. The random motions of the particles will be supposed to have a Maxwellian distribution corresponding to $T = 6000°$, though Milne (*loc. cit.*, p. 599) has shown that this is probably not exact.

Let n_+, n_e denote the number per c.c. of the ions and electrons, and V the electrostatic potential, at distance r from the centre of the cloud. In thermodynamic equilibrium at temperature T,

$$(1) \quad . \quad . \quad n_+ \propto \exp(-eV/kT), \qquad n_e \propto \exp(eV/kT),$$

so that

$$(2) \quad . \quad . \quad . \quad n_+ n_e = \text{constant} \equiv N^2,$$

where N is independent of r.

The potential V satisfies Poisson's equation, which in the present case, of spherical symmetry, has the form

$$(3) \quad . \quad . \quad . \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -4\pi e(n_+ - n_e).$$

Writing (1) in the form

$$(4) \quad . \quad . \quad . \quad \frac{d\log n_+}{dr} = -\frac{e}{kT}\frac{dV}{dr},$$

(3) may be transformed to

$$(5) \quad . \quad . \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\log n_+}{dr}\right) = \frac{4\pi e^2}{kT}\left(n_+ - \frac{N^2}{n_+}\right).$$

If n_+, n_e are nearly equal, we may, in consequence of (2), express them by

$$(6) \quad . \quad . \quad . \quad n_+ = N + \nu, \qquad n_e = N - \nu,$$

assuming that ν/N is so small that its square may be neglected. Hence (5) may be written

$$(7) \quad . \quad . \quad . \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{N}\frac{d\nu}{dr}\right) = \frac{8\pi e^2}{kT}\nu,$$

or, neglecting a small term on the left,

$$(8) \quad . \quad . \quad . \quad . \quad \frac{d^2\nu}{dr^2} + \frac{2}{r}\frac{d\nu}{dr} = \frac{\nu}{b^2},$$

where

$$(9) \quad . \quad . \quad . \quad . \quad b^2 = kT/8\pi e^2 N;$$

b has the dimensions of a length, and for $T = 6000°$,

$$(9a) \quad . \quad . \quad . \quad . \quad b = 3{\cdot}8 \,.\, 10^2 N^{-\frac{1}{2}},$$

so that even were N as small as 0·01 (which may possibly be the case when the cloud is at the earth's distance from the sun) b would not exceed 38 metres; taking $N = 10^9$ as the other extreme (possibly true near the sun), the corresponding value of b is only 0·01 cm.

The solution of (8) which is finite at $r = 0$ is

$$\nu = \nu_0 \sinh (r/b)/(r/b), \tag{10}$$

where ν_0 is the value of ν at $r = 0$. Except near this point, (10) is equivalent to

$$\nu = (\nu_0 b/2r) \exp (r/b). \tag{10a}$$

The term neglected on the left of (8) is $-(1/N)(d\nu/dr)^2$, which increases with r relative to the terms retained; but its ratio to them is of the order ν/N, so that as long as this is small, a condition already imposed in (7), the solution (10) or (10a) is approximately valid.

The solution shows that in the steady state any resultant charge which the cloud may possess is distributed so that the charge-density $2\nu e$ decreases inwards almost exponentially, at least so long as ν/N is small. Since b is small, the rate of inward decrease of ν is very rapid, and for any probable charge-density in the outer part of the cloud, the interior must be neutral to a high degree of approximation.

If the total charge Q on the cloud is not too great for ν/N to be small throughout, so that (10) is valid, then approximately

$$Q = \int_0^{r_0} (2\nu e) 4\pi r^2 dr = 8\pi e \nu' b r_0^2, \tag{11}$$

where ν' is the surface value of ν. Hence

$$\nu'/N = Q/8\pi e b r_0 = Q/(8\pi k N T)^{\frac{1}{2}} r_0^2, \tag{12}$$

and for (10) to be approximately true, Q must be such that ν'/N is small. For example, if ν'/N is not to exceed $\frac{1}{10}$, and if $r_0 = 10^9$ and $N = 10^9$, then $Q \not> 1{\cdot}4 \cdot 10^{16}$ e.s.u.; in this case, since $b = 0{\cdot}01$ cm., 99 per cent. of the charge will be situated within a surface layer less than 1 mm. thick.

The general character of the result is not altered when Q is too great for ν/N to be everywhere small. The above solution then applies except within a thin outer layer of the cloud, where ν/N may increase towards unity (positive or negative), corresponding to the almost entire absence of charges of one sign. Below this thin outer layer the gas is almost exactly neutral as before.

Naturally a strictly steady state could not be attained unless the cloud were enclosed within some envelope which prevented the dispersal of the charge on the outside. But it is useful to determine the nature of the steady state and, further, the rate at which it would be approached if, for example, the resultant charge were originally uniformly distributed throughout the cloud.

4. This rate will depend on the coefficient of diffusion (D) for the

mixture of ions and electrons. It is sufficient to calculate this from the approximate formula *

$$D = \frac{3}{n_+ + n_e}\left(\frac{\pi}{2}\frac{m_+ + m_e}{m_+ m_e}\right)^{\frac{1}{2}}\frac{(kT)^{\frac{5}{2}}}{I_1 e^4};$$

inserting T = 6000°, and numerical values for the other quantities involved, excepting N, we find

$$(13) \quad . \quad . \quad . \quad D = \frac{3{\cdot}7 \,.\, 10^{18}}{1 - 0{\cdot}048 \log N}\frac{1}{N}.$$

As N decreases, D increases rather more slowly than 1/N. The following are the values of D for several values of N :—

N	0·1	1	10	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
D	$3{\cdot}5.10^{19}$	$3{\cdot}7.10^{18}$	$3{\cdot}9.10^{17}$	$4{\cdot}1.10^{16}$	$4{\cdot}3.10^{15}$	$4{\cdot}6.10^{14}$	$4{\cdot}9.10^{13}$	$5{\cdot}2.10^{12}$	$5{\cdot}6.10^{11}$	$6{\cdot}0.10^{10}$	$6{\cdot}5.10^{9}$

The equation of diffusion within the cloud can be written in the following form, appropriate to a gas which is uniform in temperature and pressure, but non-uniform in the numbers per c.c. ($N \pm \nu$) of the two constituents, when this non-uniformity, as here, produces an electrostatic field of potential V :

$$(14) \quad . \quad . \quad . \quad \partial\nu/\partial t = D\nabla^2\nu + (NDe/kT)\nabla^2 V.$$

Using Poisson's equation, in the form $\nabla^2 V = -8\pi\nu e$, (14) becomes

$$(15) \quad . \quad . \quad . \quad \partial\nu/\partial t = D\nabla^2\nu - \nu/\tau,$$

where

$$(16) \quad . \quad . \quad . \quad \tau \equiv kT/8\pi e^2 ND = b^2/D\,;$$

τ is almost independent of the density of the cloud, since ND, by (13), varies very slightly with N over the range considered; τ has the dimensions of time, and for T = 6000°, taking ND = $5 \,.\, 10^{18}$,

$$(16a) \quad . \quad . \quad . \quad \tau = 3 \,.\, 10^{-14} \text{ second.}$$

The equation (15) is well known, and has been discussed in a variety of cases in the theory of conduction of heat. By putting $\partial\nu/\partial t = 0$, it determines the steady state, $\nu = \nu_s$, say; in the case of the spherical cloud, and when ν/N is everywhere small, ν_s is approximately given by (10). If, however, ν is originally different from ν_s, being specified by any assigned function of r, say $f(r)$, the approach to the steady state is indicated by the form of the general solution of (15), namely,

$$(17) \quad . \quad . \quad \nu = \nu_s + \{f(r) - \phi(r, t)\} \exp.(-t/\tau),$$

where $\phi(r, t)$ is a function of r and t which at time $t = 0$ equals $\nu_s + f(r)$. It is unnecessary to consider its form in detail, since the principal factor in the second term is the exponential one. This shows that the passage to the steady state is accomplished in an interval which is a moderate

* S. Chapman, *M.N.*, **89**, 55, 1928.

multiple of τ, *e.g.* 100τ. Since $\tau = 3 \cdot 10^{-14}$ second, it follows that the movement of the resultant charge in a cloud to the surface is almost instantaneous in clouds of the density considered.

The presence of a moderate proportion of neutral atoms would diminish D and increase τ, but without altering the result materially.

It may be noted that the rate of approach to the state of electrical neutrality within the cloud is practically independent of the size of the cloud. This is shown analytically by the fact that τ does not involve r_0. The electrical force at any radius r within the cloud, which determines the velocity of outward flow of charge across the sphere of this radius, depends only on the total charge within this sphere, and is entirely unaffected by any charge outside this region. For a given mean departure from neutrality within the sphere of radius r, the radial force, and consequently also the rate of flow of charge, across this sphere is proportional to r; the distance which the charge must traverse in order that the steady state may be attained is likewise proportional to r, so that the time required is independent of r.

5. It has thus been shown that whatever the resultant charge possessed by a cloud, and however this charge is initially distributed, it will almost immediately become transferred to the surface, so that the interior is practically neutral. The transfer will be effected almost entirely by flow of electrons, inward if Q is positive, and outward if Q is negative.

The charge will not remain in a thin layer at the surface, however, because this layer will begin to expand under its own repulsion. If σ is the surface density, so that

$$(18) \quad \sigma = Q/4\pi r_0^2,$$

there will be an electric force $4\pi\sigma$ at the outside of the layer, decreasing to zero on passing through the layer to the inside. The expansion of the layer will be more rapid if Q is negative than if it is positive, because in the latter case the ions, instead of the electrons, have to be accelerated.

The rate of expansion of the surface layer will be illustrated by considering the radial velocity v of any point on the spherical surface, lying within the layer, and concentric with its inner and outer surfaces, such that between it and the inner surface there lies a fraction θ ($0 < \theta < 1$) of the surface density σ. The electric intensity at this radius is $\sigma Q/r^2$, and the equation of motion of the charges at this distance from the centre of the cloud is therefore

$$(19) \quad m\dot{v} = e\theta Q/r^2,$$

or

$$(20) \quad dv^2/dr = 2e\theta Q/mr^2.$$

The solution is

$$(21) \quad v^2 = v_0^2 + (2e\theta Q/m)(1/r_0 - 1/r),$$

where v_0 is the velocity of the charges initially, when they are in the thin surface layer; the original thickness of this layer can be neglected in comparison with its initial radius r_0. A lower limit for the rate of dispersal of the layer is obtained if we write $v_0 = 0$, when the final (maximum) value of v is given by

$$(22) \quad . \quad . \quad . \quad . \quad v = (2e\theta Q/mr).$$

By writing $r = r_0 \cosh^2 u$, so that $u = 0$ when $r = r_0$, (21) can be solved in the form

$$(23) \quad . \quad . \quad r_0(\sinh u \cosh u + u) = (2e\theta Q/mr_0)^{\frac{1}{2}}t,$$

taking t to be zero when $u = 0$, *i.e.* when the layer begins to expand. From (23) we may find r for any value of t; after times long enough for u to have become large, (23) takes the approximate form

$$(23a) \quad . \quad . \quad . \quad . \quad r = (2e\theta Q/mr_0)^{\frac{1}{2}}t,$$

corresponding to the expansion of this stratum of the layer with its limiting velocity.

Comparing the values of r given by (23a) for $\theta = 1$ and $\theta = 0{\cdot}01$, it appears that 99 per cent. of the charge on the cloud finally lies within a shell of which the inner radius is $\frac{1}{10}$ the outer radius. The charge-density within this shell at radius r is $Qd\theta/4\pi r^2 dr$ or $\theta Q/2\pi r^3$, which, by (23a), is proportional to $1/r$, and therefore decreases in the ratio 10 to 1 in proceeding from the inner to the outer surface of the expanded shell; the outer charge-density is approximately two-thirds the mean density in the shell.

Suppose that k is the fractional excess of the positive charges over the negative when the cloud leaves the sun. Then

$$Q = \tfrac{4}{3}\pi r_0^3 k N e,$$

so that, by (23),

$$(24) \quad . \quad . \quad . \quad . \quad r = r_0 e(\tfrac{8}{3}\pi k\theta N/m)^{\frac{1}{2}}t,$$

showing that for given values of k, θ and N the value of r at large times t is proportional to r_0, the initial radius of the cloud.

The mean charge-density throughout the space within the outer surface ($\theta = 1$) of the expanded shell at large times t is $Q/\frac{4}{3}\pi r^3$, or, in terms of k and N,

$$(25) \quad . \quad . \quad . \quad . \quad \frac{1}{(\frac{8}{3}\pi/m)^{\frac{3}{2}}e^2(kN)^{\frac{1}{2}}t^3},$$

which is independent of r_0.

Some numerical illustrations of these formulæ will now be given. The ions will be supposed Ca^+ ions, and t will be taken as 10^5 seconds, corresponding to the time in which a cloud moving with the velocity calculated by Milne will reach the earth's orbit. Then, by (24), 99 per cent. of the excess of positive ions in the cloud will be outside a radius $1{\cdot}7 \cdot 10^6\sqrt{(kN)}$ times r_0, the initial radius; if $N = 10^9$ and $k = 1/1000$, the ratio is $5{\cdot}10^9$, far exceeding the expansion of the neutral body of the

cloud. Hence the excess charge travels far away from the cloud; this conclusion is true for any but very small values of kN.

The radius of the outer surface of the expanded shell is $5 \cdot 10^7/\sqrt{(k\mathrm{N})}$ times r_0; taking $k\mathrm{N} = 10^6$, as before, and $r_0 = 10^9$, this is $5 \cdot 10^{19}$, compared with which the radius of the earth's orbit is insignificant; the excess charge is therefore almost infinitely dispersed by the time the neutral body of the cloud reaches the earth's orbit, and this remains true even if kN is as low as unity.

The mean number of excess charges per c.c. within the space enclosed by the outer surface of the expanded layer is obtained by dividing (25) by e; it is therefore $8 \cdot 10^{-25}/\sqrt{(k\mathrm{N})}$, which is smaller the greater the initial charge-density of the cloud. Even if kN is no greater than 1, after 10^5 seconds there is only about one excess charge in a volume as large as the earth, so that they can have no appreciable effect upon the earth.

6. Whatever the sign of the initial charge on the cloud, after this charge has been dispersed the cloud will be left with a slight positive charge, because, owing to their greater thermal velocity, the cloud of electrons interspersed among the ions will tend to expand more rapidly than the cloud of ions. The positive charge adjusts itself so that the leakage of electrons is almost annulled, and the two clouds of opposite charges remain superposed. For this it is necessary that the electrostatic potential V at the surface shall approximately be given by

$$e\mathrm{V} = \tfrac{3}{2}k\mathrm{T}$$

(where k denotes Boltzmann's constant), implying that in crossing the boundary the kinetic energy of an electron is converted into potential energy. Hence for a gas at 6000°, $\mathrm{V} = 2 \cdot 5 \cdot 10^{-3}$ e.s.u. The resultant positive charge Q on the cloud is $\mathrm{V}r$, and therefore increases as the nearly neutral body of the cloud expands owing to the thermal motions and initial divergence of the cloud. The increase in Q implies that electrons slowly escape across the boundary; the rate of increase of charge Q is $\mathrm{V}v$, where v denotes the radial velocity of the boundary, which is likely to be roughly constant, and between 10^5 and 10^7 cm./sec., according to the initial divergence of the cloud. On reaching the earth's orbit, the total positive charge, taking 10^{13} cm. as an upper limit for r, will not exceed $2 \cdot 5 \cdot 10^{10}$ e.s.u. The ratio of charge to mass for the cloud, taking N and r_0 to be each 10^9, and the ions to be calcium ions, will be less than $3 \cdot 10^{-19}\, e/m_{\mathrm{H}}$.

This charge will be situated on the outside of the cloud, and there the ratio of charge to mass will be much greater than this. In § 3 the steady distribution of charge in a cloud has been found, in terms of ν, the excess of charges of one sign over the mean number per c.c. (N) of the two kinds. The solution is valid over the range for which ν/N is small; the value of this fraction is greatest at the surface, and is given by (12). Taking $\mathrm{Q} = 2 \cdot 5 \cdot 10^{-3} r$, the value corresponding to leakage of electrons, ν/N, for $\mathrm{T} = 6000°$, is about $500/\mathrm{N}^{\frac{1}{2}}r$, which is always small, since the lowest values of r that need consideration are 10^8 at

the sun and 10^{11} at the earth, while N is unlikely to be less than 10^4 at the sun and 10^{-2} at the earth. Thus ν/N is likely to be always less than $5 \cdot 10^{-8}$ for the leakage-charge layer; the corresponding ratio of charge to mass is greatest if the ions are hydrogen, when it is $(2\nu/N)e/m_H$, or $10^{-7}e/m_H$. Even this is far too small to be likely to have any importance for terrestrial problems; on the basis of analogy with Störmer's auroral calculations for a single charged particle, this maximum ratio of charge to mass in the ionized cloud is much below the value required to explain the angular radius of the auroral zone.

7. The leakage-charge layer will expand under its own repulsion, like the layer of initial charge considered in § 5. But, as Professor Lindemann has already pointed out, the velocity of expansion of the leakage-charge layer is of the same order as the thermal velocity of the ions; for, by (22), for the surface of the layer ($\theta = 1$) it is $(2eQ/mr)^{\frac{1}{2}}$, and taking $Q = \frac{2}{3}kTr/e$, this velocity is $(3kT/m_0)^{\frac{1}{2}}$, or $\sqrt{2}$ times the thermal velocity of the ions. The density of the thin charged layer will therefore decrease at a rate comparable with that of the neutral body of the cloud.

Astronomical Consequences of the Polarization of Fluorescence.
By Yngve Öhman.

(*Communicated by E. A. Milne.*)

As a result of the work of R. W. Wood, Ellet, Hanle, Keussler, and others,* the nature of the phenomena connected with the polarization of fluorescence in the absence of magnetic fields has been somewhat clarified. The spectral lines usually studied in this connection are the *Hg* $\lambda 2537$ line and the sodium D_2 line. When excited by plane polarized light and observed perpendicular to the electric vector, the fluorescence shows a polarization of 90 per cent. for the former and 16 per cent. for the latter. According to Heisenberg,† these polarization effects may be predicted from theory for the Zeeman effect in the case of observations in a strong magnetic field which is parallel to the electric vector. The intensities of the Zeeman components can be calculated by means of the ordinary intensity formulæ,‡ and the degree of polarization is given by

$$p = \frac{I_{\pi H} - I_{\sigma H}}{I_{\pi H} + I_{\sigma H}}$$

where $I_{\pi H}$ is the sum of the intensities of the π components and $I_{\sigma H}$ is half the sum of those of the σ components. The following intensity

* See P. Pringsheim, "Fluoreszenz und Phosphoreszenz," *Struktur der Materie*, vi., 1928.

† *Z.S. f. Phys.*, **31**, 617, 1925.

‡ Compare, for example, H. Hönl, "Die Intensitäten der Zeemankomponenten," *Z.S. f. Phys.*, **31**, 340, 1924.

diagrams for the above-mentioned lines were taken from an article by W. Pauli: *

The theoretical value of the degree of polarization of the *Hg* λ2537 line is easily found to be 100 per cent., since the only component that can be excited corresponds to $\Delta m = 0$. The agreement with the observed value is quite satisfactory, and the small difference may be ascribed to experimental errors.† For the D_2 line the theoretical value is 60 per cent., which is not in agreement with the observed value of 16 per cent. This result has not yet been satisfactorily explained. The D_1 line has also been investigated, and in this case no polarization has been observed. Here the facts and the theory are in complete agreement.

In the experimental investigations on the polarization of fluorescence, polarized light was always used for its excitation. If, however, the incident light is unpolarized, one may reasonably expect to observe

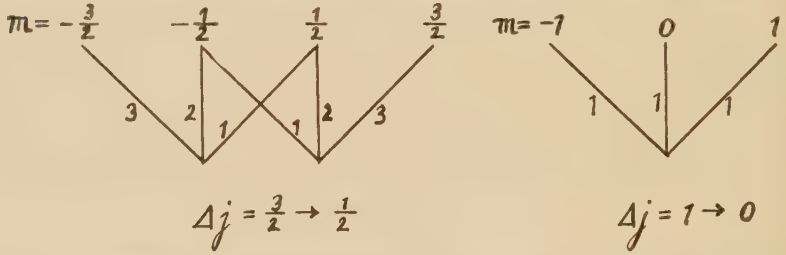


FIG. 1.—Intensity diagrams for the Zeeman Components of *Na* to the left and *Hg* to the right. The lines show the possible transitions between the magnetic quantum numbers *m*, and the numbers near the lines represent the intensities of the components on an arbitrary scale. The transitions $\Delta m = 0$ correspond to π components, $\Delta m = \pm 1$ to σ components.

polarization of the fluorescence in this case also when the observations are made perpendicular to the incident beam of light. In the case of the *Hg* line the theoretical value $p = 100$ per cent. is to be expected. For the *Na* D_2 line one would expect, in addition to the plane-polarized components, also circularly polarized ones. When analyzed, for example, by means of a Nicol's prism, the D_2 line should, according to the theory, be polarized to the extent of 43 per cent.

The aim of the present paper is to point out some eventual examples of polarization of fluorescence in celestial sources of light. Investigations of the polarization of such light have been carried out by several investigators for comets, the solar corona and nebulæ, but the aim of these investigations was to show the effects of polarization due to reflection but not due to fluorescence.

Strong reasons are given in favour of the fluorescent origin of the cometary light by Zanstra ‡ and Bobrovnikoff. § The sodium D lines

* "Quantentheorie," *Handbuch der Physik*, xxiii.
† Compare Pringsheim, *loc. cit.*
‡ *Phys. Rev.*, **28**, 428, 1926; *M.N.*, **89**, 178, 1928.
§ *Ap. J.*, **66**, 439, 1927.

have been observed several times in emission in the spectra of comets, and it seems very probable that these lines at least are produced by resonance. If such is the case, we would, according to the above discussion, expect polarization of the D_2 line but not of the D_1 line. The effect is probably not very strong, but may be of sufficient magnitude to be observed. Bobrovnikoff has already alluded to this possibility.

The bright-line spectra of the chromosphere and prominences arise in a manner analogous to fluorescence, and we have therefore reason to search for polarization in these objects. In the neighbourhood of the sun's limb we have, of course, little hope of finding any such effects, since the incident light is not parallel but originates from half a sphere. At greater distances from the limb the conditions are, however, more favourable, and it seems very probable that such effects might be found in the high prominences. The Balmer lines and the H and K lines of ionized calcium are the most conspicuous ones observed in these objects. The Balmer lines show the normal Zeeman triplet, and should, therefore, give a polarization of 100 per cent. when excited by parallel light. The Ca^+ lines show Zeeman effects analogous to those in the case of the sodium lines, and we would therefore expect to find K faintly polarized and H unpolarized. No reports of observations made on the polarization of light from high prominences have been found in the literature. It would be of considerable physical interest if such an investigation were carried out.

The polarization of the light from the solar corona has been established and studied in the case of several solar eclipses, but the observations were almost always made on the continuous spectrum only. In 1900, R. W. Wood * examined the polarization of the bright-line spectrum, and the observations, which were made visually, seemed to show that the lines were unpolarized. It is difficult to give theoretical values for the polarization of the coronium lines at the present time. It is true, according to Ira Freeman,† that most of the lines appear to belong to argon; but, since the spectra of this element are very complicated, one must be careful in applying the ordinary intensity rules for the Zeeman components.

In connection with the suggested polarization effects in the neighbourhood of the sun it may be worth while to recall the facts regarding its general magnetic field. In laboratory experiments on fluorescence it has been found that the polarization is often influenced by magnetic fields, and indeed even by the weak field of the earth. According to the investigations carried out at the Mount Wilson Observatory,‡ the general magnetic field of the sun is about one hundred times as strong as the field observed at the surface of the earth. It seems, however, to decrease very rapidly with increasing height. At great distances from the surface the magnetic field has perhaps no influence on the fluorescence.

Investigations on the polarization of light from nebulæ have been

* *Publications of the U.S. Naval Observatory*, **4**, D116, 1905.
† *Ap. J.*, **68**, 177, 1928.
‡ *Contributions from the Mount Wilson Observatory*, p. 302, 1925.

carried out by Meyer * at the Lick Observatory. For the planetary nebulæ he could find no effects indicating polarization. According to modern views the light from a planetary nebula seems to be of fluorescent origin. Since the Balmer lines are often very prominent in nebular spectra, it is necessary, from the point of view adopted in the present paper, to explain the absence of polarization. The explanation is, no doubt, that the lines appearing in the spectra of the planetary nebulæ are probably not due to resonance. According to Zanstra † and Bowen,‡ the temperature of the central star in such a nebula seems to be about 50,000°. Most of the atoms in the nebula are then ionized photo-electrically by absorption of radiation in the extreme ultraviolet, and very few atoms absorb the Balmer lines. If a proton encounters an electron, the electron falls from one energy level to another by successive steps, thus producing the spectral lines; one would, therefore, not expect to find polarization of the Balmer lines. It would, however, be of some interest to investigate this point experimentally for each of the spectral lines rather than for the total photographic region, as was done by Meyer. It does not, however, seem very likely that polarization effects will be observed in the case of the planetary nebulæ.

In connection with the polarization of fluorescence there is an interesting question having to do with the formation of solar absorption lines. The *Hg* $\lambda 2537$ line, when excited by a parallel unpolarized beam of light, is, as mentioned above, polarized to the extent of 100 per cent. when it is observed perpendicular to the direction of the incident beam. On the other hand, the polarization is zero in the direction parallel to the incident light. From this fact it may be concluded that the intensity of the fluorescent radiation parallel to the incident beam is twice as great as that perpendicular to it. If, therefore, there were no magnetic field about the sun, and accordingly no disturbance of the polarization of fluorescence, the radiative transfer would be affected, since the emission per gram of matter, which is ordinarily put equal in all directions, would depend on the distribution of the intensity of the incident light with direction. In the atmospheric layers of the sun, where the inward flow of energy is very faint, a slight modification of the radiation process would result.

* *Lick Observatory Bulletin*, No. 328, 1920.
† *Ap. J.*, **65**, 50, 1927; *Nature*, May 19, 1928.
‡ *Ap. J.*, **67**, 1, 1928.

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The Hydrogen Chromosphere. By W. H. McCrea.

(*Communicated by R. H. Fowler, F.R.S.*)

1. *Introduction and Summary.*—Calcium, in an ionised state, and hydrogen are the elements observed at greatest heights in the sun's chromosphere and most conspicuous in its " flash " spectrum. Theory has so far been largely concerned with the calcium, and its physical condition is known through Professor Milne's researches. This paper deals with the hydrogen, and we find that converging lines of evidence point to the rather surprising conclusion that the pressure, density law, and means of support are of a nature quite different from those of the calcium.

We are in the first place concerned to give an interpretation of the published observations at least sufficiently accurate in orders of magnitude to give direction to, and to check the validity of, possible theories. In fact it appears that we cannot go much beyond orders of magnitude until we have an adequate theory.

The outline of the content is :

i. We obtain from the absolute measurement of H_γ the absolute density of emitting atoms and its variation with height. We deduce tentatively that there are about 10^{12} normal atoms per cm.^{3} at the base of the chromosphere. This is a rather larger number than we shou have expected.

ii. We interpret observations of the continuous spectrum at the head of the Balmer series to mean that there are about 2×10^{10} ionised atoms per cm.^{3} in the same region. With certain assumptions about ionisation under such conditions, we find this result also leads to an estimate of about 10^{12} normal atoms.

These numbers are to be compared with 10^6 Ca^+ ions given by Milne's theory.

iii. The relation of this excess of hydrogen to that in the reversing layer suggested by other writers is remarked upon.

iv. It is shown that the number of free electrons in the ionised gas is just that required to give an intensity of scattered light of the order actually observed for the base of the corona. Not knowing the density law of the *ionised* gas we cannot make a complete comparison, but it strongly suggests itself that the corona is part of the same atmosphere of ionised hydrogen.

v. Ionisation in the chromosphere is discussed. It appears that it must be of the order of magnitude of that in thermodynamic equilibrium at a temperature of the same order as that defined by the other properties.

This agrees with observations for hydrogen.

vi. It is shown that Ca^+ under the conditions usually assumed would probably be ionised to Ca^{++}, but that the excess of ionised hydrogen prevents this.

vii. The possibility of a " pressure effect " is briefly discussed.

viii. We examine the problem of mechanical equilibrium. The cal-

culation mentioned in (i.) indicates that only one-quarter of the total weight can be supported by gas-pressure in an isothermal atmosphere. No modification of gravitational equilibrium is found adequate. Hence radiation-pressure is investigated. It is found that the black body radiation of the sun cannot support one-tenth of the weight. No success is found with possible modifications of either the radiation or the behaviour of the atoms with respect to it.

Only electrical forces, apparently, remain to be tried. Their importance in this connection is suggested by Rosseland's theory of the excess of hydrogen in the outer layers of a star.

It is hoped to examine these later, but this section seems to show definitely that the usual factors supposed effective in chromospheric equilibrium are not so in the case of hydrogen.

ix. The possibility that they are effective in other stars is briefly discussed.

2. *The Interpretation of the Observations.*—The most recent and extensive observations of the atmospheric hydrogen are those of Davidson and Stratton, of Davidson, Minnaert, Ornstein, and Stratton, and of Pannekoek and Minnaert.*

The authors of Ref. C have measured the absolute intensity of the H_γ hydrogen line in the "flash" spectrum and its variations with height above the sun's limb. They find their results can be expressed by saying that 1 cm.^{2} of the chromosphere at a height y cm. above the limb radiates †

$$6{\cdot}2 \times 10^5 \exp(-1127y/a) \text{ ergs per unit solid angle per sec.} \quad . \quad (2.01)$$

Here a is the solar radius $= 6{\cdot}951 \times 10^{10}$ cm.

Suppose provisionally there is no appreciable reabsorption of this radiation in its passage through the chromosphere. Then the exponential factor shows that it must virtually all come from close to the sun, and so the light we observe from 1 cm.^{2} may be taken as that emitted in a column of 1 cm.^{2} section in the direction of observation.

Consider such a column whose nearest point to the limb is distant $y(= a\eta)$ from it. Let $x(= a\xi)$ be distance measured along the column from this point.

If the density of emitting atoms at η is $n_0 \exp(-1127\eta)$, then the number in the column is

$$2n_0 a \int_0^\infty \exp\{-1127(\sqrt{(1+\eta)^2+\xi^2}-1)\}d\xi \quad . \quad (2.02)$$

$$= 2n_0 a \exp(-1127\eta) \int_0^\infty \exp\left(-1127\frac{\xi^2}{2}\right)d\xi$$

$$= \sqrt{\frac{2\pi}{1127}}\, n_0 a \exp(-1127\eta) \quad . \quad . \quad . \quad . \quad (2.03)$$

* Davidson and Stratton, *Mem. R.A.S.*, **64**, 105, 1927; Davidson, Minnaert, Ornstein, and Stratton, *M.N.*, **88**, 536, 1928; Pannekoek and Minnaert, *Verh. d. kon. Akad. Amsterdam*, xiii., No. 5, 1928. These papers will be referred to as Ref. A, Ref. B, Ref. C, respectively.

† Actually this is their result when reduced to the above units.

expanding the expression in brackets in (2.02) as far as terms in ξ^2, η^2, since it is found that the only important part of the integral depends on small values of these quantities. This is on account of the large numerical factor 1127.

The result (2.03) shows that, on account of the orders of magnitude involved, we are justified in taking the density as obeying the same exponential law as the observed intensity. This is what Pannekoek and Minnaert do, and they point out that the law is $\exp\left(\frac{-Hgy}{4kT}\right)$ if T = 5100° K, H being the mass of the hydrogen atom and g the value of gravity at the surface, indicating that gas-pressure must only balance a quarter of gravity at this temperature. This last result takes the number of atoms in the emitting state as a constant fraction of the whole. But, for the order of pressures considered, this fraction can evidently not increase as we recede from the sun, and excludes the possibility, on this account, of a larger coefficient for $-y$. It also, in taking the mean mass of the particles as H, neglects ionisation; but to this we shall later return.

For $\eta = 0$, the radiation per unit solid angle by the number of atoms given by (2.03) by transitions to state $l = 2$, *i.e.* in emitting $H_\gamma(=\nu_5^2)$ is

$$\frac{1}{4\pi}\sqrt{\frac{2\pi}{1127}}an_0A_5^2h\nu_5^2 \text{ ergs per sec.} \quad . \quad . \quad (2.04)$$

where A_5^2 is the probability coefficient for this process.*

Substituting numerical values (2.04) becomes

$$4{\cdot}72 \times 10^3 n_0 \text{ ergs sec.}^{-1}.$$

Comparing with (2.01), this gives

$$n_0 = 132. \quad [\text{Note } n_0 = (n_5)_0.] \quad . \quad . \quad (2.05)$$

I have attempted to show (*l.c.*) that the distribution of the atoms among the stationary states may be expected to imitate thermodynamic equilibrium at temperature T′ (say). T′ is found to be such that, given n_1, $n_2(T') = \frac{1}{2}n_2(T)$, where T is the temperature of the sun's radiation, supposed black body radiation. If T = 6000° K, we find T′ = 5800° K. (In *l.c.* I took too low a value for T, but the general result cannot be very different for T = 6000°.)

Taking this result, the number of atoms per cm.^{3} in the Balmer state ($l = 2$) at $\eta = 0$ will be

$$(n_2)_0 = \tfrac{4}{25}(n_5)_0 \exp(h\nu_5^2/kT') = 6200 \quad . \quad . \quad (2.06)$$

Suppose the line H_γ in absorption to have mean width 1A. Then the calculated value of A_5^2 gives an atomic absorption coefficient for

* A value for A_5^2 is given by the present writer in *Proc. Camb. Phil. Soc.*, **24**, 506, 1928, to which we shall refer as *l.c.* We shall use the notation of *l.c.*, viz. total quantum number = $l = 1, 2, 3, \ldots$; number of atoms in state $l = n_l$; energy difference between states, $l, l_1 = h\nu_l^{l_1}(l > l_1)$; probability for spontaneous transition, $A_l^{l_1}$. The upper suffixes are to be distinguished from exponents.

this line about 7×10^{-15}. So 6200 atoms per cm.^{3} will give an absorption factor about $\exp(-4 \times 10^{-11})$ per cm. or $\exp(-4 \times 10^{-6})$ per km. So, considering the exponential decrease in density, we see that reabsorption must be very small. Our calculations neglecting it are therefore valid.

In the same way we expect to have

$$(n_1)_0 = \tfrac{1}{2}\tfrac{1}{5}(n_5)_0 \exp(h\nu_5{}^1/kT') = 1{\cdot}03 \times 10^{12} \quad . \quad (2.07)$$

Comparing this with the number 10^6 (approx.) for Ca^+ ions on Milne's theory, we see how surprisingly large it would be for the chromosphere.

Now the Balmer lines are seen, and so we actually see the atoms going into state 2. So we are fairly confident about the order of n_2 in (2.06). Further, in absorption we see them leaving this state and can get another measure of the number in it, but probably this time lower down in the reversing layer. We have, on the other hand, no means of "seeing" the state 1, and so are a little chary about accepting such a large estimate as (2.07).

We ask, Is there no other way of seeing that there must be something like this number of particles present? It turns out that there is, and in the following way. It must be emphasised that the procedure is only intended to give an order of magnitude, but we believe it does so satisfactorily.

In Ref. A, Davidson and Stratton have distinguished the separate Balmer lines up to about $l = 40$, in which neighbourhood they begin to shade into the continuous spectrum. Now it is observed that the intensity in the separate lines in this neighbourhood is about the same as that of the continuous spectrum where it meets the head of the series. From Ref. C we know the absolute intensity of H_γ, and from Ref. B * we know that the intensity from line to line can be calculated approximately from the results of *l.c.* This then gives an estimate of H_{38} (*i.e.* $l = 40$). Then with any reasonable assumed mean width of H_{38} we get an estimate of the absolute intensity in the continuous spectrum.

In Ref. B, Table II. shows that the order of magnitude of the line intensities can be obtained by setting $T' = 4000°$ K, and we shall take this to calculate n_{40} from n_5 in order not to get any serious overestimate.†

The total intensity emitted per cm.^{3} in H_{38} is then

$$n_{40}A_{40}{}^2h\nu_{40}{}^2 = \frac{6{\cdot}88 \times 10^9}{40^5}\left(\frac{40}{5}\right)^2 n_5 h\nu_{40}{}^2 \exp\left(\frac{-h\nu_{40}{}^5}{kT'}\right) \quad . \quad (2.08)$$

where $A_{40}{}^2$ is obtained from equation (1.71) of *l.c.*, and n_5 is known from (2.05).

* These measurements cover the range $l = 6$ to $l = 28$, and for our purpose it should be allowable to extrapolate. This being so, the estimate of the continuous intensity is really empirical and does not depend on the *theory* of *l.c.*

† The table quoted shows that 4000° is too low for I_l/I_{l+1} if l is small, and too high if it is large, and so it must be fairly good for comparing large l with small l.

To calculate the intensity of the continuous spectrum in terms of the number of free hydrogen nuclei, N_+, and free electrons, N_-, we make use of Kramers's target area for capture into a χ-level, of an electron with velocity v, viz. apart from a numerical coefficient not very different from unity,*

$$\beta = \frac{32\pi^2 e^6}{3\sqrt{3}c^3 m^2 h\nu v^2}\Delta\nu \quad . \quad . \quad . \quad (2.09)$$

where

$$\tfrac{1}{2}mv^2 = h\nu - \chi \quad . \quad . \quad . \quad . \quad (2.10)$$

and where m, e are the electronic mass, charge.

Applying this to $\chi^{(2)}$, the energy of the Balmer state, Kramers's $\Delta\nu$ is

$$\Delta\nu^{(2)} = \frac{4\chi^{(2)}}{h}\left(\frac{1}{(2-\frac{1}{2})^2} - \frac{1}{(2+\frac{1}{2})^2}\right) = \left(\frac{16}{15}\right)^2 \frac{\chi^{(2)}}{h}.$$

The number per cm.^{3} of captures of electrons with velocities between v and $v + dv$ is therefore

$$4\pi N_+ N_- \left(\frac{m}{2\pi k T_1}\right)^{\frac{3}{2}} e^{-\frac{1}{2}mv^2/kT_1} \beta v^3 dv = N_+ N_- f dv \text{ (say)} \quad . \quad (2.11)$$

assuming the free electrons have a Maxwellian distribution of velocities given by a temperature T_1.

We have to decide on a value of T_1. We shall take 5000° as being the temperature usually assumed for molecular velocities in this region. (If the determining factor were the velocity of photo-electric ejection, we should expect, by analogy with *l.c.*, the 5800° estimate to be nearer the truth, but an underestimate of T_1 merely means an underestimate of N_+, N_-.)

By (2.10) these collisions result in emissions in the range

$$d\nu = \frac{\nu^2}{c}|d\lambda| 10^{-8} = \frac{mv}{h}dv \quad . \quad . \quad . \quad (2.12)$$

where $d\lambda$ is measured in angstroms. We shall take $N_+ = N_-$, which will be allowed if it turns out that N_- is largely due to hydrogen. Hence the intensity emitted per cm.^{3} per angstrom is

$$10^{-8} N_+^2 f \cdot \frac{h\nu^2}{cmv} \cdot h\nu \quad . \quad . \quad . \quad . \quad (2.13)$$

Let ν now correspond to the head of the series, ν_0, so that

$$\chi = h\nu_0, \qquad v = 0.$$

Then (2.13) reduces to

$$9{\cdot}91 \times 10^{-8} \frac{e^6 h\nu_0^3}{c^4}\left(\frac{\pi}{mkT_1}\right)^{\frac{3}{2}} N_+^2 \quad . \quad . \quad . \quad (2.14)$$

If we take the width of H_{38} to be 1A, then observation, as already

* Kramers, *Phil. Mag.*, **46**, 836, 1923.

explained, shows that (2.08) and (2.14) must be about equal. Setting them equal, we obtain finally

$$N_+ = 1{\cdot}9 \times 10^9 \sqrt{n_5} = 2{\cdot}2 \times 10^{10} \quad . \quad . \quad (2.15)$$

putting in the value of n_5 for the base of the chromosphere from (2.05).

This in itself is about 10^4 times the number of Ca^+ ions supposed present in the same region. It is independent and direct evidence of a large excess of hydrogen in this region. It does not depend in a very sensitive manner on the particular values attributed to T′, T_1.

Now there is nothing to prevent these ions and electrons also forming normal atoms. Since the ions are evidently not removed by gravity as soon as formed, it is not likely the normal atoms will be. So we conclude the ionised atoms must be in a steady state with respect to a number of normal atoms, which, were there thermodynamic equilibrium, would be given by the usual ionisation formula. We discuss shortly, in a later paragraph, ionisation under these conditions, but here let us suppose it is given as regards order of magnitude by a "temperature" T_2 of the order of those defined by other properties. Let us take 5500° as being fairly high and not likely to give a large overestimate of n_1.

Then *

$$n_1 = N_-^2 \left(\frac{h^2}{2\pi mkT_2}\right)^{\frac{3}{2}} \exp(hRc/kT_2) \quad . \quad . \quad (2.16)$$

since the I.P. of normal hydrogen is hRc, where R is Rydberg's constant = 109,700.

This gives

$$n_1 = 2{\cdot}45 \times 10^{-9} N_-^2 = 1{\cdot}2 \times 10^{12} \quad . \quad . \quad (2.17)$$

for N_- given by (2.15).

This is exactly the order given by (2.07). The pressure of normal atoms given by (2·17) is

$$8{\cdot}2 \times 10^{-7} \text{ atmospheres} \quad . \quad . \quad . \quad (2.18)$$

for a temperature of 5000° K.

Both estimates, of course, are derived from the same estimate of n_5, but in quite different ways. We thus have added confidence in the order of magnitude found for n_1, and in both methods of calculation, and, what is important, this lends support to our assumption that the ionisation is of the order given by thermodynamic equilibrium at a reasonable temperature.†

Should the inference of the large number of normal atoms be doubted,

* See, for example, Eddington, *Internal Constitution of the Stars*, p. 251, 1926.

† A shorter way of getting at these results, but one which may not exhibit the steps in the argument so clearly, is as follows: The amount of light *emitted* per angstrom in the continuous spectrum at ν_0 = the amount which would be *absorbed* by ($l = 2$) atoms if there were thermodynamic equilibrium. This can be calculated by Kramers's *absorption* coefficient. Also, the amount emitted in H_{38} would equal the amount absorbed in H_{38}. It is found that these are nearly equal, practically independently of temperature. This, then, checks the fact that the orders of magnitude are those for thermodynamic equilibrium. Hence the numbers of normal and ionised atoms can be calculated.

it must be emphasised that the large number of ionised atoms is a fact of observation. The latter may prove the more interesting result.

Lest it be considered that we are founding too much upon measurements of a single line, it may be mentioned as a check that Taylor's * theoretical intensities for Ca^+ are found to be of the same order of magnitude as those observed for this line.

3. *Other Evidence.*—It is well known that other lines of investigation indicate a surprisingly large predominance of hydrogen in the outer layers of the sun. The independent methods of Miss Payne † and Unsöld ‡ both do so. Their results presumably refer to slightly lower layers than we are considering, but if there is there such an excess, it is not so strange to find it in the chromosphere as well.

The present result goes a little further, in exhibiting the excess apart from the number of atoms in the normal state.

4. *Relation to the Corona.*—It is interesting to see how much light will be scattered by the free electrons of the ionised gas.

Consider first 1 cm.^{3} containing $N(r)$ electrons distant $r(= a\zeta)$ from the centre of the sun. Let I_0 be the intensity of the solar radiation. We can take the classical scattering coefficient since we are dealing with optical frequencies. § We shall not make a large error if we take the scattering as averaged over all directions, especially since the incident radiation is spread over a large solid angle.

Then the intensity dI of the scattered radiation is

$$dI = \frac{4\pi}{3}\frac{N(\zeta)e^4}{m^2c^4}\left\{1 - \sqrt{1 - \frac{1}{\zeta^2}}\right\}I_0 \quad . \quad . \quad (4.1)$$

The factor in brackets measures the departure from 2π of the solid angle subtended by the sun at distance r. But, since the important values of r are near a, there will be no more error in taking this factor as unity than there must be in our estimate of $N(\zeta)$. Approximations for it can easily be introduced into the subsequent integrations if necessary.

Hence, by the arguments at the beginning of § 1 the intensity from 1 cm.^{2} is approximately

$$\begin{aligned}&\frac{4\pi}{3}\sqrt{\frac{2\pi}{1127}}N_0\frac{ae^4}{m^2c^4}I_0\exp(-1127\eta)\\ &= 1{\cdot}7\times10^{-15}N_0I_0\exp(-1127\eta)\\ &= 3{\cdot}7\times10^{-5}I_0\exp(-1127\eta) \quad . \quad . \quad . \quad (4.2)\end{aligned}$$

for N_0 given by (2.15).

The value $3{\cdot}7\times10^{-5}I_0$ for the base of the corona is of the right order of magnitude. For example, Kienle || gives as an upper observational estimate, $10^{-5}I_0$.

* *M.N.*, **87**, 616, 1927.
† *Stellar Atmospheres* (1925), Table 28.
‡ *Zs. f. Phys.*, **46**, 765, 1928.
§ See, for example, Birtwistle, *New Quantum Mechanics*, p. 193, 1928.
|| *Göttingen Nachrichten*, p. 89, 1928.

Further, we can estimate the total amount of light scattered, assuming the effective part of the layer is thin. For in that case it forms a screen with $N_0 a \int_0^\infty \exp(-1127\eta)d\eta = N_0 a/1127$ electrons per cm.^{2}. This scatters a fraction *

$$\frac{8\pi}{3 \times 1127} \frac{ae^4}{m^2c^4} N_0 = 9 \times 10^{-7} \quad . \quad . \quad . \quad (4.3)$$

of the light incident on it. This again is of the right order of magnitude, but too small by a factor of perhaps 2.

It strongly suggests itself, therefore, that the continuous light of the corona is scattered by electrons produced by the chromospheric hydrogen (accompanied, of course, by the corresponding positive ions).

We have been using the exponential law given by observations of H_γ. It would evidently be far too condensed for the corona. But it must be remembered that this law measures the decrease in ($l = 5$) atoms. Clearly the decrease in ionised atoms must be much slower, since the ionisation will increase, and the fractional concentration of ($l = 5$) atoms consequently decrease, as we recede from the sun's surface to lower pressures.

Just to see the kind of effect, suppose the chromosphere were in isothermal thermodynamic equilibrium. Then $n_1 \propto n_5$. But the ionisation formula gives

$$(N_-)^2 \propto n_1.$$

Hence

$$N_- \propto \sqrt{n_5} \propto \exp(-564\eta) \quad . \quad . \quad . \quad (4.4)$$

which would also give approximately the law of decrease of light intensity.

This is not so sudden, but is still vastly too sudden to agree with the facts. In any case, however, we cannot expect the simple type of exponential law to hold for a spherical atmosphere where η is no longer small compared with unity. In fact, in Ref. C the authors remark that the decrease of H_γ itself appears less rapid at the greatest heights considered. Clearly further investigation is necessary, but it is bound up with the general problem of the equilibrium of the hydrogen, to be discussed below. At any rate we have found an unexpected quantity of ions and electrons in the chromosphere, and seen that these prove to be just sufficient to give the order of the observed continuous intensity there. So we anticipate that the outer corona is directly connected with these ions and electrons.

We may mention here that this scattering gives rise to a radiation-pressure about 2×10^{-24} dynes per electron. This is small compared with the force of gravity even on the electrons alone.

5. *Ionisation in the Chromosphere.*—We seek next to form some theoretical estimate of the ionisation.

As Milne has more than once emphasised in this connection, we may

* Eddington, *Internal Constitution of the Stars*, p. 77.

only do so in these conditions of non-thermodynamic equilibrium from a comparison of rates of ionisation and recombination (capture).

Consider first the rate of capture into the normal state.

The quantity $\Delta\nu$ in (2.09) for this case is

$$\Delta^{(1)}\nu = \frac{\chi^{(1)}}{h}\left(\frac{1}{(1-\frac{1}{2})^2} - \frac{1}{(1+\frac{1}{2})^2}\right) = \frac{32}{9}\frac{\chi^{(1)}}{h} \quad . \quad (5.1)$$

so that

$$\beta^{(1)} = \frac{32^2}{27\sqrt{3}}\frac{\pi^2 e^6 \chi^{(1)}}{c^3 m^2 h^2 \nu v^2} = \frac{\kappa\chi^{(1)}}{\nu v^2} \text{ (say)} \quad . \quad . \quad (5.2)$$

Hence the total number of captures in state 1 per cm.^{3} per sec. is

$$4\pi N_+ N_- \left(\frac{m}{2\pi k T_1}\right)^{\frac{3}{2}} \kappa\chi^{(1)} \int_0^\infty e^{-\frac{1}{2}mv^2/kT_1} \frac{v}{\nu} dv \quad . \quad . \quad (5.3)$$

The integral can be written :

$$\frac{h}{m} e^{\chi^{(1)}/kT_1} \int_{\nu_1}^\infty e^{-h\nu/kT_1} \frac{d\nu}{\nu} \simeq \frac{he^{h\nu_1/kT_1}}{m\nu_1} \int_{\nu_1}^\infty e^{-h\nu/kT_1} d\nu$$

$$= \frac{kT_1}{m\nu_1} = \frac{hkT_1}{m\chi^{(1)}} \quad . \quad . \quad . \quad . \quad (5.4)$$

since $e^{-h\nu/kT_1}$ decreases rapidly for $\nu > \nu_1$, the head of the Lyman series, for any T_1 we shall use.

The quantity (5.3) is then, to this approximation,

$$2\kappa h \left(\frac{m}{2\pi k T_1}\right)^{\frac{1}{2}} N_+ N_- = \mu_1 \text{ (say)} \quad . \quad . \quad . \quad (5.5)$$

Let now B be the *total* target area for capture on the classical theory. Then *

$$B = 2^{\frac{1}{2}}\pi^{\frac{7}{5}} \frac{e^4}{m^2 c^4} \left(\frac{c}{v}\right)^{\frac{14}{5}} = \frac{\lambda}{v^{\frac{14}{5}}} \text{ (say)} \quad . \quad . \quad . \quad (5.6)$$

giving for the quantity corresponding to (5.3),

$$4\pi N_+ N_- \left(\frac{m}{2\pi k T_1}\right)^{\frac{3}{2}} \lambda \int_0^\infty e^{-\frac{1}{2}mv^2/kT_1} v^{\frac{1}{5}} dv \quad . \quad . \quad (5.7)$$

A numerical approximation to the integral gives $0{\cdot}74\left(\frac{2kT}{m}\right)^{\frac{3}{2}}$, and so (5.7) becomes

$$\frac{2{\cdot}96}{\sqrt{\pi}} \lambda \left(\frac{m}{2kT_1}\right)^{\frac{7}{10}} N_+ N_- = \mu \text{ (say)} \quad . \quad . \quad . \quad (5.8)$$

We then find that for $T_1 = 5000°$ K,

$$\frac{\mu}{\mu_1} = 1{\cdot}04 \times 10^4 \quad . \quad . \quad . \quad . \quad (5.9)$$

It is readily found that μ_l for the next few values of l is of the same order as μ_1.

* Eddington, *loc. cit.*, p. 224.

Thus the large ratio (5.9) means that if an electron combines with an ion with the emission of radiation, the probability that it goes straight into a low orbit (l small) is relatively very small. Applying the result to thermodynamic equilibrium, we see that the number of photo-ionisations per unit time from orbits with $l > l'$ is large compared with the number from $l<l'$ where l' is small (say, for definiteness, 5). This is precisely analogous to the result found in *l.c.* that the number of atoms entering states of large l depends immediately on the radiation bringing them from neighbouring states and not from states of small l.

In the present problem the ionisation depends immediately on the ionisation by low frequency radiation from large quantum orbits, and not on the very high frequencies acting on the excess of normal atoms. That is to say, ionisation by radiation is to be thought of as a step-by-step process, the radiation exciting the atom by stages into a high orbit and then ionising it.*

Then, further, on account of the importance of low frequencies, we expect there to be some process of "stimulated capture" exceeding in importance "spontaneous capture" for these frequencies. This is from analogy with line absorption and emission, and may mean that ionisation and capture in high orbits take place much more frequently, even, than we have estimated.†

Now consider what these results mean for the chromosphere.

The rate of ionisation depends on the number of atoms in high orbits and the intensity of the radiation in comparatively low frequencies.

The rate of recombination depends on the number of ions and electrons and on the "temperature" of the motion of the electrons, but also on the same intensity of radiation.

We do not know without much investigation what "temperature" the motion of the electrons will have, but we expect it to be of the same *order* as the temperature of the incident radiation. For one thing, we are dealing with a region comparatively near the source of radiation, but also, if the determining factor is the velocity of ejection from the atoms, we expect it to be the temperature of the incident radiation even if the latter is contained in part only of the whole solid angle.

Hence, in the balance, only the quality and not the absolute intensity of (*i.e.* not the solid angle containing) the incident radiation matters as far as orders of magnitude of ionisation go.‡ If the relative numbers of atoms in high levels correspond to some temperature also of the order of that of the incident radiation, we expect that, relative to these numbers, the number of ionised atoms must do so also.

This has supposed the main factor in ionisation to be radiation. Should it turn out to be collisions our result would still be true, supposing

* From the point of view of a single atom. Statistically, of course, in thermodynamic equilibrium the separate processes must balance their reverse processes.

† Compare the fact that in *l.c.* (2.01) the expressions are $\diagdown\!\!\diagup A_l^{l_1}$ for small $\nu_l^{l_1}$.

‡ Since it is acting both ways—ionising atoms and stimulating their recombination.

the temperatures of mean motion and of distribution among orbits to be of the same order. In other words, we conclude that the different properties, atomic motion, distribution among the stationary states (*l.c.*), degree of ionisation, will define temperature parameters of similar orders of magnitude.

This is a crude theory to justify the approximate trustworthiness of our calculations above for hydrogen.

6. *Application to Calcium.*—Apart from all approximations to equilibrium conditions, we conclude that the rate of ionisation of Ca^+ to Ca^{++} estimated by Milne * must be multiplied by a factor about 10^3 or 10^4. He has only considered ionisation from the normal state. It consequently seems impossible that the ions should be removed by gravity as soon as formed. Under the conditions contemplated by him we should expect the Ca^+ to be very quickly entirely re-ionised.

Under the present conditions, however, there are present near the base of the chromosphere 10^{10} or more free electrons per cm.^{3} due to the hydrogen. This gives a Ca^{++} ion a tremendously increased chance of picking up an electron again.

We can easily estimate the effect of this, for Ca^+ and neutral hydrogen have roughly the same I.P. But the latter, in the presence of these 10^{10} electrons per cm.^{3} (or rather $2{\cdot}2 \times 10^{10}$ in the actual calculation), we have seen to be only about 2 per cent. ionised under conditions we have reason to believe prevail there. Hence, if the calcium also approaches anything like equilibrium with respect to these electrons, it can also be ionised to approximately only the same degree, on account of its similar I.P., independently of its own absolute concentration.†

Since the density of free electrons must diminish with height, we should anticipate that some more appreciable ionisation of the Ca^+ to Ca^{++} should set in at greater heights.

The theory of the calcium chromosphere seems to demand some such mechanism for controlling the second-stage ionisation. In fact, if our views hold good, the usual type of "chromospheric equilibrium" is only possible in the presence of a sufficient number of free electrons.

At the risk of labouring the point, we must stress the fact that §§ 5, 6 only deal with orders of magnitude. For instance, we can conceive of the ionisation being appreciably reduced by a lower temperature of mean motion of the electrons rendering them an easier prey; or by the importance of "spontaneous captures" being comparable with "stimulated captures," giving the capturing processes an advantage under reduced radiation. We do not see, however, that these will upset the orders of magnitude.

7. *The Possibility of a Pressure Effect.*—One further point that is

* *M.N.*, **88**, 188, 1928.

† This idea was inspired by a discussion with Professor Franck. Owing to the small first I.P. of *Ca* we do not expect the first-stage ionisation to be appreciably affected. *Added in proof*: Taking equal I.P.'s gives unfortunately rather too rough an estimate. If the Ca^+ (I.P. 11·82 volts, compared with hydrogen 13·55 volts) were in thermodynamic equilibrium with the $2{\cdot}2 \times 10^{10}$ electrons per cm.^{3} at 5500° it would be 41 per cent. ionised to Ca^{++}, but if at 5000° only 4·7 per cent. ionised.

more likely to find an explanation if these larger pressures are present is the falling off in intensity of the higher Balmer lines indicated by the observations in Ref. B.

The energy steps diminish, while the linear dimensional steps increase from l to $l+1$, with increasing l. Now the indicated effect increases with l. We expect it, therefore, to depend in some way on the size of the orbits and not so much on radiative properties. The only way the size can come in is in collisions, *i.e.* a pressure effect. A pressure of 10^{-6} instead of 10^{-13} (Ca^+) atmospheres would be more likely to show it.

Taking Urey's * theory and his *numerical* values, we find the pressure is still much too low to show his effect. In Ref. B it is also stated that were such an effect present it would come in too suddenly as a function of l to agree with the facts. But it may be pointed out that the light observed comes from layers of different pressure, a circumstance which would tend to smooth down this suddenness.

8. *The Problem of Mechanical Equilibrium.*—Having attempted an interpretation of the facts, we turn now to survey the problem of the mechanical support of the hydrogen. We must examine in turn the various possibilities.

First consider gas-pressure. Isothermal equilibrium at temperature T gives a density varying as $\exp\left(\frac{-Hga\eta}{k\mathrm{T}}\right)$ for small η. This is for small ionisation. For large ionisation the mean mass of the particles approaches $H/2$, and the density law $\exp\left(\frac{-Hga\eta}{2k\mathrm{T}}\right)$. Neither of these agrees with observation. The effect of varying T—for example, an adiabatic law or $\mathrm{T} \propto r^{-1}$, r^{-2}—may be tried, but the same excessive concentration to very small η remains.

The densities we have seen reason to suppose present mean that the free paths of even the neutral atoms are small compared with the dimensions of the chromosphere. Even if, however, this were not true, one can see no reason why calculations based on the ordinary gas laws should not hold good.

Ordinary gas-pressure fails, therefore, as might be expected, to give an explanation. So also does any explanation depending on the presence of other gases. Ca^+ is the only gas we should expect to help, but it seems impossible that the radiation-pressure on it should support also forty times its weight of ionised hydrogen, let alone all the neutral hydrogen.

Second, consider radiation-pressure. We begin with line absorption.

In *l.c.* we have seen that if the sun radiates like a black body, the distribution of the atoms near the sun among the energy levels is like that at a temperature somewhat lower than the radiation. Hence the effect of the lines is only overestimated by attributing to the atoms the temperature of the radiation and taking the same number in the normal state. We can do this and express the radiation-pressure of the

* *Ap. J.*, **59**, 1, 1924.

full radiation as a double sum over all initial and final l's in absorption switches.

On account of the final negative result it is not worth giving the rather lengthy working here. It is eventually found that the first Lyman line gives a pressure of higher order than all the others combined. The temperature of the radiation being taken as 6000° K, this pressure is found to be

$$3{\cdot}04 \times 10^{-21} n_1 \text{ dynes per cm.}^3 \quad . \quad . \quad . \quad (8.1)$$

The force of gravity on the total number n of atoms per cm.^{3} is

$$45{\cdot}5 \times 10^{-21} n \text{ dynes per cm.}^3 \quad . \quad . \quad . \quad (8.2)$$

Since, of course, $n_1 < n$, the conclusion is that, under these conditions, radiation-pressure cannot balance $\frac{1}{10}$th of gravity.

The general investigations of Eddington and Milne have shown that in the chromosphere continuous absorption cannot be an important means of support. This can be checked directly from their values of the absorption coefficients and any reasonable assumed densities of ions and electrons. One can also try Rayleigh scattering on the classical theory, but one finds it provides no appreciable pressure.

One next looks for modifications of radiative equilibrium arising from either (*a*) a modified behaviour of the atoms, or (*b*) a modification in the incident radiation. Take these in turn.

(*a*) The Balmer lines are in an intense part of the spectrum, but our investigation shows they are ineffective as a means of support because of the excessive number of atoms in the normal state. Could the Balmer state ($l = 2$) for any reason behave as a normal state, there might be an adequate radiation-pressure. This is practically the suggestion of Compton and Russell,* who point out that of the 2_2, 2_1 Balmer states the latter is metastable, and that the consequent longer "life" of this state may account for the anomalous behaviour of stellar hydrogen.

The existence of such a state does not, however, prevent the excess in the normal state. If the method of *l.c.* is applied to, say, an atom with three levels of which the second is metastable, one finds that the relative numbers in the first two states are just those in equilibrium at the temperature of the radiation.† The magnitudes of the A coefficients show that such a state would be established exceedingly rapidly.

There is evidently no solution in this direction, and so we turn to (*b*).

(*b*) When we supposed the sun to radiate like a black body at 6000°, we found that virtually any radiation-pressure there was was due to the first Lyman line. This is far in the ultra-violet, and the black-body assumption may fail in this region. Were this radiation to correspond to an appreciably lower temperature, its pressure would become negligible. If, however, it were to correspond to, say, 7500°, the

* *Nature*, **114**, 86, 1924.

† See also a footnote on this point in Rosseland, *M.N.*, **85**, 541, 1925.

pressure per normal atom would be greatly increased. After the failure of all previous suggestions, this hypothesis seems at first sight rather attractive.

Neglecting, for the moment, ionisation, and applying the method of Milne,* it is found that this assumption leads to a density law

$$n = n_0 \exp\left(\frac{-\mu Hgy}{kT}\right) \quad . \quad . \quad . \quad (8.3)$$

where μ is the coefficient of partial support and here can take any value in $1 \geqslant \mu > 0{\cdot}16$, T is the temperature of mean motion; and

$$n_0 = 1{\cdot}2 \times 10^7 \mu \frac{(\mu - 0{\cdot}16)}{(1-\mu)} \Delta\lambda \quad . \quad . \quad . \quad (8.4)$$

for T = 5000. Here $\Delta\lambda$ is the width of the line in angstroms.

If $\mu = \frac{1}{4}$ we get the right exponential law, but then $n_0 \cong 4 \times 10^5 \Delta\lambda$. Since $\Delta\lambda$ is of the order unity, n_0, and so n, is vastly too small.

So although the hypothesis has apparently qualitative success it quite fails quantitatively, and there is no point in discussing its likelihood. In addition, the qualitative success can only be apparent, since we should expect the gas under such a low pressure to become practically completely ionised.

We have thus reached the rather surprising conclusion that the hydrogen chromosphere is supported by neither gas nor radiation-pressure. The only remaining type of force one can suppose to act on the gas is electrical.

That electrical forces may be important is suggested by the mere fact that we have so much hydrogen present. For, as Unsöld† says, this seems to be a verification of Rosseland's‡ theory whereby electrical forces in a star tend to send the hydrogen to the surface. This, however, is due to the presence of other elements, and an examination of Rosseland's equations seems to show that in the region occupied mostly by hydrogen, the ordinary simple gas-law equations should give the density distribution. Nevertheless, his work shows that the possibility of electrical forces should be taken into account, and his simple equations may have to be modified near the star's boundary.

What is apparently demanded is an examination of the electrical state of the solar atmosphere, and the author hopes to attempt this.

9. *Other Stars.*—It will be noticed that for hotter stars or for giant stars, radiation-pressure may greatly exceed gravity for normal atoms. But our considerations show that ionisation will prevent extended chromospheres being formed under this pressure. One would think the bright hydrogen lines of an O-type star might arise in a chromosphere of such an extent that there was in it a wide departure from equilibrium with the temperature of the star's photosphere. It appears, however, that the very high temperature of the radiation

* *M.N.*, **85**, 111, 1924; **86**, 8, 1925.
† Unsöld, *loc. cit.*
‡ Rosseland, *loc. cit.*

would maintain such a high degree of ionisation that if there were such a chromosphere it must be supported in some other way than by radiation-pressure.

I am very grateful to Professor Franck and to Professor Kienle for opportunities to discuss this work with them.

Göttingen :
1929 *January.*

Stellar Parallaxes determined Photographically at the Cape Observatory. (*First List.*) By H. Spencer Jones and J. W. Jackson.

A programme of stellar parallax determinations was commenced at the Cape Observatory in 1926 April. In the present paper are given the first results derived from this programme.

The photographs are obtained with the Victoria telescope, aperture 24 inches. The scale, as determined from five scale plates, is 1 mm. = 15·31726 minutes of arc. The magnitude of the parallax star is reduced by means of an adjustable rotating sector. A series of gauges are used to adjust the opening of the sector to definite degrees of magnitude reduction, proceeding in steps of 0·5 magnitude. The sector, which is mounted on hardened pivots running in jewelled bearings, is driven by an air blast from a fine nozzle supplied by a small air compressor. A rapid rotation can in this way be secured without the least trace of vibration, the shutter having been so constructed that it is in perfect balance whatever the opening.

The comparison stars in the mean are of the eleventh magnitude. Under good conditions an exposure of $2\frac{1}{2}$ minutes is sufficient, but longer exposure becomes necessary when the definition is inferior. Three exposures are given on each plate, the telescope being displaced in declination between successive exposures.

The programme of observation comprises mainly stars with proper-motions exceeding 30″ per century, south of Dec. − 10°. A few stars north of this limit have been included in order to secure comparisons with parallaxes determined at other observatories. Many of the stars are faint. The 18-inch visual telescope on the same mounting is used for guiding ; an 8-inch guiding telescope, with eye-piece mounted on adjustable scales, is also attached to the mounting. The early observations were confined to stars of intermediate magnitude, neither bright enough to require local desensitisation of the film nor faint enough to require an increased length of exposure.

Measurement of the plates has been made in a manner similar to that employed at Greenwich. After suitable comparison stars have been selected, a reference plate of selected plate glass is ruled with fine lines in the positions of the comparison stars. The photographs of any one region are measured against the same reference plate, the ruled face of the reference plate being placed against the film and measurement

made through the reference plate. The two plates are adjusted before being placed in the measuring machine so that the distance between any star and the corresponding ruling on the reference plate is less than one revolution of the micrometer screw, and are then held together by small clips.

The plates are measured in one of the Repsold micrometers. The periodic errors of the micrometer screw have been investigated and found to be negligible. With the method of measuring employed, the progressive errors do not enter into consideration. Each plate is measured both in the direct and reversed position. The number of comparison stars used is in general four, five, or six, and averages about five ; in a few instances it has been necessary to use three or seven stars.

The measurements have been reduced by Schlesinger's convenient method of dependences. The measures of each exposure have been reduced separately, and in many cases the direct and reversed measures have also been reduced separately. One of the advantages of Schlesinger's method is that this can be done with very little additional computing.

It has been found necessary to weight the plates in the reductions according to the quality of the images, dependent upon the atmospheric conditions at the time of exposure. The average definition at the Cape is poor, especially in the summer months ; the definition for each plate is entered by the observer in the observing book. The weight of the plate is assessed by the measurer at the time of measurement according to the quality of the images. Provisional weights were assigned in the first instance and used for a preliminary reduction ; from the residuals for the first fifty regions reduced it was found that the provisional weights were relatively too high for the poorer plates, and the following weights have been finally adopted :—

Images. / Exposures.	Good.	Fair.	Poor.
One	·50	·30	·20
Two	·80	·50	·25
Three	1·00	·60	·35

The fifty regions have been reduced, again employing these weights, so that the whole material should be uniform in the method of weighting.

The observations of each star extend over at least five epochs. They are only carried on to a sixth or seventh epoch if the weight is insufficient for a satisfactory parallax determination. With the exception of No. 67 (ζ Telescopii), all the parallaxes contained in the present list are based upon observations at five epochs. The observations of ζ Telescopii were extended to a sixth epoch. The number of plates for a single determination averages 22, but the total weight (with unit weight corresponding to one good plate with three images) is appreciably smaller. The weight of the individual parallaxes varies between about 7 and 12.

The average probable error for a plate of unit weight is $\pm$ 0″·025.

The probable errors assigned to the parallaxes and proper-motions in the following list are based upon the probable error of unit plate derived from the internal agreement of the series. This probable error is given so that the probable errors corresponding to the average error for unit plate of ± 0″·025 can be inferred.

In the following list the Durchmusterung number of the star is given as in the Draper Catalogue, *i.e.* the B.D. number down to Dec. − 23°, the Co. D. number south of this limit to − 52° and the C.P.D. number south of − 52°. The magnitudes and spectral type are from the Draper Catalogue. The first value for the proper-motion is that given by meridian or other observations ; the data have been extracted from various sources and some of the proper-motions are very uncertain, being based upon the comparison between two catalogues only or upon approximate measures made with the blink-microscope. The second value is the relative proper-motion in R.A. (the direction in which the plates have been measured) derived from the photographs. In the last column, the first value of the probable error given for any star refers to the parallax and the second value to the proper-motion.

The parallaxes and proper-motions in R.A. derived from the photographs are relative values. The mean magnitude of the comparison stars may be taken as 11^m.

The plates from which the parallaxes in this list have been derived were measured by Messrs. Spencer Jones, Jackson, Peirce, Turner, and Miss Wilkin.

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900).	Proper-Motion. R.A. Unit ″·001.	Proper-Motion. Dec. Unit ″·001.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001.
				h m s					
1	−44° 52	7·6	Go	0 12 14	+429	− 60	24	− 3	± 8
		8·8		−44° 24′·5	+349				± 8
2	−27° 108	7·8	G5	0 19 20	+668	+ 66	23	+ 50	± 8
		8·7		−27° 35′·0	+661				± 8
3	78 G Sculp.	6·6	Go	0 28 50	− 55	−521	31	+ 45	±11
	−35° 170	7·4		−35° 32′·4	− 63				±11
4	−49° 141	6·8	Go	0 31 50	+340	−140	22	+ 34	± 7
		7·8		−49° 40′·8	+373				± 7
5	−31° 325	7·2	G5	0 48 6	+630	+ 28	18	+103	± 6
		8·0		−30° 54′·1	+544				± 7
6	−16° 162 (*p*)	7·9	..	0 53 11	..	..	29	+ 39	±10
				−16° 13′·5	+ 3				±10
	−16° 162 (*f*)	7·7	..	0 53 11	..	..	19	+ 36	± 7
				−16° 13′·4	− 5				± 7
	Mean	7·0	F5	..	..	..	..	+ 37	± 6
		7·4			− 2				± 6

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900). h m s	Proper-Motion. R.A. Unit ″·001.	Dec.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·00
7	−30° 549	8·1	Ko	1 32 30	+324	−110	18	+ 52	± 6
		8·9		−29° 54′·1	+253				± 6
8	−23° 693	8·9	K5	1 48 2	+858	+ 7	34	+ 81	±11
		10·7		−22° 55′·7	+841				±11
9	−64° 156	7·6	Go	2 7 26	+337	0	19	+ 4	± 7
		8·0		−64° 49′·6	+340				± 6
10	Boss 2199	6·0	G5	8 13 39	+280	−991	22	+ 81	± 7
	−12° 2449	6·8		−12° 17′·6	+301				± 8
11	−39° 4574	6·4	Go	8 33 42	−357	0	27	+ 56	± 9
		6·7		−39° 47′·9	−301				±10
12	−38° 4789	6·6	Ko	8 39 35	−317	+310	26	+ 94	± 8
		7·3		−38° 31′·9	−323				±10
13	88 G Hydræ	5·9	Go	8 54 2	+244	+215	31	+ 21	±10
	−15° 2656	6·5		−15° 45′·1	+251				±11
14	−25° 6661	7·2	F8	8 57 31	−276	+272	27	+ 24	±10
		8·3		−26° 10′·7	−244				±10
15	−47° 4831	7·3	G5	9 15 31	−233	+190	23	− 8	± 9
		8·0		−47° 33′·1	−150				± 8
16	−46° 5238	8·6	Ko	9 27 17	+385	−384	21	0	± 8
		9·0		−46° 56′·0	+454				± 9
17	−35° 5751	6·0	Ko	9 27 23	+164	−248	22	− 7	± 8
		7·0		−35° 16′·2	+168				± 8
18	175 G Hydræ	5·0	Go	9 37 44	−410	+236	28	+101	± 9
	−23° 8646	5·6		−23° 28′·0	−407				±10
19	−42° 5678	8·4	K5	9 47 4	+488	−488	27	+ 89	± 9
		8·4		−43° 1′·3	+475				± 9
20	−40° 5610	8·4	F5	9 50 58	−275	+170	26	+ 12	±10
		8·6		−40° 17′·4	−289				± 9
21	−19° 2926	7·2	Go	10 3 38	−130	−330	25	+ 18	± 9
		8·2		−19° 15′·4	−106				±10
22	−23° 9258	7·0	Go	10 19 34	−346	+ 76	19	+ 31	± 7
		7·5		−24° 6′·1	−342				± 8
23	−26° 7942	8·8	Ko	10 24 0	+331	−253	28	+ 24	±10
		9·6		−26° 50′·8	+348				±11

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900). h m s	Proper-Motion. R.A. Unit ″·001.	Dec.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001.
24	−58° 2243 (*p*)	..	..	..	..	..	23	+ 8	± 9
					+148				±10
	−58° 2243 (*f*)	..	..	..	..	..	31	− 6	±12
					+142				±13
	Mean	8·8	Ko	10 25 16	+198	−150	..	+ 3	± 7
		9·8		−58° 16′·7	+146				± 8
25	236 G Hydræ	5·9	F8	10 31 34	+266	−684	27	+ 64	±10
	−11° 2918	6·4		−11° 41′·6	+239				±11
26	−14° 3199	8·1	Go	10 45 31	−290	− 80	27	+ 25	± 9
		8·7		−14° 34′·3	−291				± 9
27	−30° 8807	8·2	Go	10 49 29	−146	−353	24	+ 39	± 7
		9·2		−30° 37′·1	− 92				± 8
28	−14° 3263	8·3	Go	11 2 24	−265	+182	30	+ 29	±10
		8·9		−14° 49′·4	−199				±10
29	−14° 3277	9·3	K5	11 6 29	+712	−580	26	+ 56	± 9
		10·5		−14° 25′·9	+678				± 9
30	−32° 7996	6·9	F8	11 12 48	+ 59	−333	23	+ 19	± 7
		7·4		−32° 59′·5	+ 80				± 8
31	−58° 3496	7·9	Ko	11 16 30	−310	0	26	− 6	± 9
		8·1		−58° 9′·5	−323				± 9
32	−60° 2911	7·4	K5	11 20 20	−507	+ 70	38	+ 65	±13
	(mass)	9·0		−61° 6′·1	−502				±13
33	289 G Hydræ	6·1	G5	11 29 38	−676	+813	19	+ 89	± 6
	−32° 8179	6·8		−32° 18′·1	−694				± 6
34	294 G Hydræ	6·5	Go	11 36 10	−326	+203	29	+ 21	±10
	−28° 9027	6·9		−28° 38′·9	−313				±12
35	−57° 4948	8·4	Ko	11 38 40	0	+390	23	+ 27	± 8
		9·0		−57° 27′·7	− 38				± 9
36	−65° 1723	8·5	Ko	11 43 19	−254	+200	27	− 2	± 9
		9·5		−65° 33′·7	−186				±10
37	−23° 10243	8·7	Ko	11 47 44	−247	+195	15	+ 4	± 5
		9·3		−23° 55′·3	−228				± 5
38	−38° 7479	6·6	F5	11 58 32	−370	− 37	29	+ 35	± 9
		7·7		−38° 27′·0	−379				±10

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900).	Proper-Motion. R.A. Unit ″·001.	Proper-Motion. Dec.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001
				h m s					
39	−11° 3246	6·8	Go	12 4 18	+304	−161	32	+ 27	±10
		7·3		−11° 17′·7	+302				±11
40	−40° 7288	10·0	Go	12 20 17	−434	+180	32	+ 12	± 9
		9·8		−40° 17′·9	−256				±10
41	−13° 3557	9·6	K2	12 28 51	−482	− 25	28	+ 26	± 8
		10·7		−14° 5′·2	−497				± 9
42	−43° 7953	6·0	Go	12 49 24	−243	245	21	+ 40	± 6
		7·6		−43° 36′·2	−220				± 7
43	−51° 7244	9·0	K5	12 59 15	−805	−840	26	+ 38	± 8
		10·0		−51° 53′·1	−792				±10
44	−21° 3660	7·3	G5	13 4 19	+163	−360	25	+ 73	± 8
		8·3		−21° 39′·0	+200				±10
45	−40° 7824	6·7	F8	13 14 16	−346	− 11	22	+ 5	± 7
		7·3		−40° 39′·9	−311				± 8
46	−23° 11071	8·4	Ko	13 21 13	−362	− 57	29	+ 56	±10
		9·7		−23° 46′·3	−342				±11
47	−23° 11076	8·6	Ko	13 21 37	−307	− 65	17	+ 8	± 6
		9·7		−23° 46′·1	−312				± 6
48	−26° 9740	7·2	F8	13 23 0	−273	+195	27	+ 22	± 8
		7·7		−26° 52′·8	−206				± 9
49	−58° 5131	9·7	..	13 38 54	−413	0	23	+ 28	± 7
				−58° 8′·8	−480				± 9
50	−35° 9019	6·5	Go	13 43 10	−539	−170	26	+ 4	± 8
		7·2		−35° 12′·0	−510				±10
51	−56° 5956	8·2	G5	13 43 39	−190	−250	24	+ 4	± 7
		8·8		−56° 45′·2	−297				± 9
52	−74° 1142	6·0	Go	13 59 16	−238	−270	21	+ 37	± 6
		6·6		−74° 22′·8	−248				± 7
53	−11° 3684	7·9	G5	14 6 25	−259	−185	19	+ 6	± 6
		8·7		−12° 8′·2	−194				± 7
54	−58° 5467	7·0	Ko	14 12 0	−426	−880	18	+112	± 6
		8·2		−58° 53′·7	−449				± 7
55	−63° 3246	7·6	Go	14 17 8	−352	−280	29	− 1	± 9
		8·2		63° 14′·9	−258				±10

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900). h m s	Proper-Motion. R.A. Unit ″·001.	Dec.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001.
56	Boss 3734	6·2	F8	14 31 41	−872	+364	30	+ 25	±10
	−11° 3770	6·7		−11° 52′·8	−851				±10
57	−23° 11940	7·7	K2	14 46 0	−927	−431	26	+ 54	± 7
		8·6		−23° 52′·8	−963				± 9
58	Boss 3812	8·9	Ma	14 51 36	+950	−1664	19	+148	± 6
	−20° 4123	10·1		−20° 57′·7	+988				± 7
59	−48° 9607	8·5	Ko	14 59 14	− 59	−550	24	+ 7	± 7
		9·4		−48° 52′·7	− 19				± 8
60	−27° 10236	7·6	A3	15 3 5	+286	− 98	24	+ 12	± 8
		8·4		−27° 58′·8	+ 36				± 9
61	−8° 3949	7·9	F8	15 14 37	+ 88	−192	27	+ 34	± 9
		8·4		− 8° 17′·6	+110				± 9
62	−16° 4112	8·3	G5	15 28 18	− 88	−319	29	+ 20	± 9
		9·1		−16° 39′·4	−109				±10
63	−59° 6206	6·1	F5	15 31 46	−121	−270	25	+ 34	± 7
		6·4		−59° 34′·5	−114				± 9
64	−37° 10500	6·1	Go	15 40 59	−450	−268	26	+ 80	± 9
		6·9		−37° 36′·0	−421				± 9
65	−26° 12862	7·5	Go	18 3 2	− 66	−319	23	+ 17	± 7
		8·0		−26° 7′·3	− 10				± 8
66	−63° 4370	6·2	Go	18 10 0	+125	−270	23	+ 13	± 7
		6·7		−63° 54′·9	+ 14				± 8
67	ζ Telescopii	4·1	Ko	18 21 8	+144	−257	33	+ 38	±10
	−49° 12153	5·9		−49° 7′·5	+128				±10
68	α Scuti	4·1	Ko	18 29 46	− 22	−317	19	− 8	± 6
	−8° 4638	5·1		− 8° 18′·8	− 18				± 8
69	−25° 13291	7·4	Go	18 31 1	+163	−279	28	+ 51	± 9
		8·2		−25° 44′·8	+148				±10
70	−56° 9037 (*p*)	..	..	..	..	..	39	+ 50	±12
					+ 34				±13
	−56° 9037 (*f*)	..	..	..	..	..	25	+ 65	± 9
					+ 65				±10
	Mean	8·7	K5	18 49 12	0	−500	..	+ 60	± 7
		10·1		−56° 6′·5	+ 54				± 8

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900).	Proper-Motion. R.A. Unit ″·001.	Proper-Motion. Dec. Unit ″·001.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001
				h m s					
71	−5° 4811	8·2	G5	18 50 34	−191	−369	21	+ 14	± 6
		9·0		−5° 51′·8	−190				± 8
72	−12° 5278	7·1	F8	19 0 52	−186	−379	27	+ 9	± 9
		7·6		−12° 2′·3	−162				±11
73	−66° 3425	8·2	K2	19 12 38	..	..	31	+ 11	± 9
		9·3		−66° 21′·5	− 24				±11
74	−66° 3427	8·6	Ko	19 12 56	− 24	−390	29	− 3	± 9
		9·6		−66° 19′·7	− 22				±11
75	−66° 3431	7·9	K2	19 15 19	−250	+265	20	+ 36	± 6
		9·0		−66° 40′·1	−226				± 7
76	−33° 14164	7·5	Go	19 16 23	+ 50	−440	34	+ 22	± 9
		8·0		−33° 5′·8	+103				±11
77	Boss 4985	7·0	Go	19 26 26	+ 61	−739	20	+ 25	± 6
	−28° 15936	7·9		−28° 12′·7	+ 72				± 8
78	−59° 7505	7·3	Go	19 34 1	+314	−240	21	+ 37	± 6
		8·5		−59° 14′·2	+141				± 7
79	86 G Telescopii	6·4	F8	20 14 25	−396	−258	24	+ 42	± 7
	−50° 12929	7·1		−50° 18′·5	−324				± 8
80	−58° 7755	9·1	Go	20 21 29	+393	0	20	+ 34	± 7
		9·4		−58° 17′·0	+273				± 7
81	−31° 17597	6·7	Go	20 21 32	− 39	−550	24	+ 58	± 7
		7·3		−31° 10′·9	− 7				± 9
82	−21° 5840	8·0	G5	20 43 28	+256	−247	21	+ 17	± 7
		8·6		−20° 59′·7	+217				± 8
83	119 G Pavonis	5·8	Go	20 58 54	+409	−331	22	+ 28	± 7
	−73° 2192	6·5		−73° 33′·9	+482				± 9
84	−14° 5936	7·2	Ko	21 1 37	+372	− 23	20	+ 51	± 7
		8·2		−14° 19′·4	+396				± 8
85	−13° 5945	9·4	K5	21 24 30	+1019	−263	20	+ 58	± 7
		10·6		−12° 56′·4	+1039				± 9
86	−51° 12998	7·2	Ko	21 29 50	+366	−200	18	+ 83	± 6
		7·9		−51° 17′·2	+400				± 6
87	−27° 15550	6·7	Go	21 33 19	+361	−101	21	+ 51	± 7
		7·6		−27° 45′·3	+387				± 7

No.	Name and D.M. No.	Mag. Vis. Phot.	Type.	R.A. Dec. (1900). h m s	Proper-Motion. R.A. Unit ″·001.	Proper-Motion. Dec. Unit ″·001.	P.E. Unit Weight. Unit ″·001.	Parallax. Unit ″·001.	P.E. Unit ″·001.
88	−58° 7893	8·6	Ma	21 39 24	0	−902	28	+ 52	± 9
		10·2		−58° 8′·2	+ 78				±10
89	−69° 3204	8·8	G5	21 46 58	−384	−250	24	+ 22	± 8
		9·6		−69° 29′·2	−176				± 8
90	−45° 14576	8·3	Ko	22 0 29	+324	−280	18	+ 27	± 5
		9·5		−45° 52′·3	+366				± 6
91	−51° 13248	9·1	G5	22 16 17	+ 28	−1580	20	+ 23	± 6
		9·3		−51° 16′·9	+175				± 7
92	−49° 13852	7·6	G5	22 21 10	+309	−350	38	− 6	±12
		7·5		−49° 51′·9	+350				±12
93	−28° 17861	8·0	Go	22 32 35	+439	− 10	23	+ 21	± 7
		8·4		−27° 57′·8	+436				± 7
94	−43° 15148	9·7	G5	22 40 11	−296	− 70	32	− 12	± 9
		10·4		−42° 56′·8	−211				±11
95	−42° 16092	8·1	G5	22 44 29	−390	−260	26	+ 45	± 8
		8·4		−42° 0′·8	−353				± 9
96	−36° 15693	7·4	Map	22 59 25	+6780	+1281	28	+241	± 9
		9·0		−36° 25′·7	+6737				±11
97	−51° 13489	9·2	Ko	23 5 17	−121	−330	29	+ 24	± 9
		10·0		−51° 30′·2	− 24				±10
98	Boss 6042	6·4	K2	23 24 22	+172	−226	26	− 9	± 9
	−5° 5999	7·5		−5° 4′·6	+177				±10
99	−66° 3799	7·3	F8	23 44 1	+354	0	29	+ 28	± 9
		7·8		−66° 48′·7	+325				±11
00	−10° 6206	7·8	Go	23 52 22	+433	−134	30	+ 13	±10
		8·4		−10° 12′·0	+447				±10
01	−26° 16876	9·2	A2	23 56 43	−148	−380	15	+ 9	± 5
		9·3		−26° 20′·8	−159				± 6

On the Probable Errors of some Stellar Radial Velocities observed at the Cape of Good Hope and at Mt. Hamilton and Santiago. By W. W. Campbell, Associate R.A.S.

At the meeting of the Royal Astronomical Society on 1928 June 8, and in the *Monthly Notices R.A.S.* for June,* two interesting papers were presented by Dr. H. Spencer Jones, H.M. Astronomer at the Cape of Good Hope, entitled "The Orbits of Four Long-Period Spectroscopic Binaries," and "The Radial Velocity Variations of α Orionis and α Scorpii." The observations utilized by Dr. Jones in preparing these papers were secured exclusively by the Cape Observatory and by the Lick Observatory (at Mt. Hamilton and at Santiago, Chile).

My comments relate chiefly to a minor conclusion reached by Dr. Jones in the second of the papers, to the effect that the probable error of a Lick observation (of unit weight) of α Orionis or of α Scorpii is decidedly larger than the probable error of a Cape observation; a result, Dr. Jones says, in speaking of α Orionis, that is "in accord with what would be anticipated from" the greater dispersion of the spectrograph employed at the Cape. Although his conclusion is based upon a large number of observations—242 made at the Cape and 132 at Mt. Hamilton and Santiago—I am not convinced that the probable errors he assigns to a Cape observation and to a Lick observation are correct; and as his remarks describe, to the best of my knowledge, the only attempt thus far made to compare the accuracy of observations of this kind obtained at the two observatories, it seems not inappropriate to examine with some care the premises and argument by which he reached his conclusion. A recent brief respite from administrative duty has permitted me to do this.

Dr. Jones's values of the probable error of one Cape observation and of one Lick observation are reached not by any of the usual methods of determining probable error, but by a procedure devised by himself, apparently for the purpose of obtaining some idea of the magnitude of the irregularities which occur in the apparent radial velocities of the star; that is, of departures from elliptic orbital motion. That such departures exist for α Orionis, α Scorpii, and certain other stars has, of course, long been known; and they are of sufficient magnitude to suggest that orbital motion is not the underlying, perhaps not even the partial, cause of the more or less periodic variations in their observed velocities. This is clearly recognized by Dr. Jones in many parts of his paper. Nevertheless, the laws of ordinary orbital motion, supplying as they do readily calculable periodic curves with a number of arbitrary parameters, provide a systematic means for approaching the study of stars that exhibit periodic variations of radial velocity; and they have not infrequently been used for that purpose as Dr. Jones is here using them. It should be observed, however, that the assumption of orbital motion in such cases cannot, without great caution, be made the basis for evalu-

* **88,** 644–679.

ating the accuracy of observations. The method of least squares provides dependable values of probable error only when the law of variation is accurately known ; a fact that will be universally admitted, and is stated here only to indicate the direction of the leading steps in this criticism.

After discussing briefly the irregular or, as Dr. Jones terms them, the " short-period " fluctuations of α Orionis, as to its apparent velocity, Dr. Jones set himself the task of determining their general order of magnitude ; and for this purpose he made the tacit assumption that these fluctuations follow the so-called law of errors. From this he derived the equations in the following group, in which

σ_C designates the probable error of a Cape observation,
σ_L ,, ,, ,, Lick ,,
σ ,, " probable error due to fluctuations."

The third and fourth equations in the group represent evaluations of the probable errors of a Cape and a Lick observation respectively, made by comparing consecutive observations secured within short intervals of time, during which intervals any short-period fluctuations in velocity were regarded by him as negligible. The equations are :

$$\left.\begin{aligned} \sigma_C^2 + \sigma^2 &= 0{\cdot}53 \\ \sigma_L^2 + \sigma^2 &= 0{\cdot}98 \\ 2\sigma_C^2 &= 0{\cdot}55 \\ 2\sigma_L^2 &= 0{\cdot}61 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

" whence," to quote Dr. Jones,

$$" \sigma_C^2 = 0{\cdot}23, \quad \sigma_L^2 = 0{\cdot}37, \quad \sigma^2 = 0{\cdot}46 " \quad . \quad (2)$$

By extracting the roots, he reached the conclusion that

$$\left.\begin{aligned} \sigma_C &= \text{P.E. of one Cape observation of } \alpha \text{ Orionis} = \pm 0{\cdot}48 \text{ km./sec.} \\ \sigma_L &= \text{P.E. ,, Lick ,, ,, ,,} = \pm 0{\cdot}61 \text{ ,,} \\ \sigma &= \text{P.E. due to irregular fluctuations} = \pm 0{\cdot}68 \text{ ,,} \end{aligned}\right\} \quad (3)$$

The significance of the " whence " is not entirely clear, and it would have been helpful if the derivation of the quantities (2) had been more specifically set forth. Relations (1) form a group of four simultaneous equations containing three unknown quantities, the values of two of which are explicitly defined. It is curious to read in (1) that $2\sigma_C^2 = 0{\cdot}55$, and directly below that $\sigma_C^2 = 0{\cdot}23$; also, in (1) that $2\sigma_L^2 = 0{\cdot}61$, and right below that $\sigma_L^2 = 0{\cdot}37$. That some form of adjustment was necessary is obvious, and a statement of its nature seems essential to an understanding of Dr. Jones's conclusions. His sole comment relative to this really critical matter is that " The equations are not entirely consistent, but this was hardly to be expected, as the Lick plates are distributed differently in time from the Cape plates, and the effect of short-period fluctuations will not be the same for the two series." With the quoted statement I cordially agree ; but

I incline to think Dr. Jones will join me in saying that his adoption of one and the same symbol (σ) for the "probable error due to fluctuations" in both the Cape and Lick observations is equivalent to assuming that the effect of short-period fluctuations *is* "the same for the two series"; in other words, equivalent to assuming that the irregularities we are calling short-period fluctuations repeated themselves, *in effect*, in all of the cycles covered by the observations. The contents of the following paragraphs should be of interest in this connection.

α Orionis has long been known as a star irregularly variable in brightness. Plummer at the Lick Observatory in 1907 discovered the variability of its observed radial velocity. Since 1910 as to the radial velocity, and since 1913 as to the brightness, it has been known that the period of variation is approximately six years, and that the curves of observed radial velocity and of observed brightness do not go on repeating themselves in successive periods.

Of spectrograms of α Orionis there had been secured—

at Mt. Hamilton,	5	with Orig. Mills spectrograph	3-pr., 16-in.	camera.
„ „	56	„ New „ „	3-pr., 21-in.	„
in Chile,	5	„ Chile spectrograph	3-pr., $21\frac{1}{4}$-in.	„
„ „	17	„ „ „	2-pr., 16-in.	„
at the Cape,	141	„ Cape „	4-pr., 30-in.	„

Dr. Jones plotted all the Cape and Lick observations, and was led to an approximate period of 5·79 years. The 224 single observations were then plotted in one cycle of this length; though it should be said that one of the Lick observations, made on 1919 November 28, was used as of date one year later. The plotted points for the 83 Lick observations were found to lie, on the average, 0·39 km./sec. higher than those for the 141 Cape observations. Accordingly, quoting Dr. Jones, "in order to bring the Lick measures into the best general agreement with the Cape measures, a correction of $-0{\cdot}39$ km./sec. has been applied to the published Lick velocities." Unfortunately, by some oversight, this correction was applied by Dr. Jones only to the first 51 Lick observations and not to the last 32; and with these partially corrected results the normal places were formed and the least-squares solution for elliptic orbital elements carried through.

The 141 Cape observations of α Orionis extend from 1903·88 to 1913·12, but excluding the first two, which were separated by two years from the third observation of the series, they lie between the time limits 1906·12 and 1913·12; that is to say, 139 observations are distributed through only seven years, or *1·21 cycles of variation.*

The Lick observations of α Orionis, 83 in all, extend from 1896·76 to 1926·17; that is, through 29·4 years, or *5·1 cycles.* Forty-one of the Lick plates lie within the time interval, 1906·12 to 1913·12, but the other 42 Lick plates are distributed through four other cycles. Now, if the velocities of α Orionis were known to repeat themselves precisely, cycle after cycle, this "averaging" procedure, and the later application of the method of least squares to the determination of an orbit, could in

some degree be justified; but such repetition in successive cycles appears not to be a fact. Further, the 139 + 41 = 180 observations have been super-powerful, under the method-of-least-squares treatment, in forcing the deduced orbit—save as to the period of revolution—to fit best the much observed 1·21 central cycles; and the other 2 + 42 = 44 observations, because of the failure of the observed velocity curve to repeat itself in the different cycles, have been in some degree reduced in effective weight. The 7 Lick observations made in the seven years before the Cape series of 141 observations began, and the 32 Lick observations made in the thirteen years after the Cape series ended, have not exerted the same relative individual influence upon the results of the least-squares solution that they would have if the Cape observations had been distributed through the 29·4 years in somewhat the same way as the Lick observations.

That the velocity curve does not always repeat itself in the successive cycles may be seen from the following data:—

The first five Lick observations, made with the Original Mills spectrograph in the interval 1896 October to 1898 February, are all *below* Dr. Jones's velocity curve; 0·9, 1·5, 2·3, 1·1, and 1·8 km./sec. below the curve. The mean of the five residuals is −1·52 km./sec., at mean phase 3·97 years.

The last six observations of α Orionis, secured with the New Mills spectrograph in the interval 1925 December to 1926 March, are all *above* Dr. Jones's curve; 3·0, 1·8, 2·6, 1·6, 1·5, and 1·6 km./sec. above the curve. The mean of the six residuals is + 2·02 km./sec., at mean phase 3·77 years.

Thus, the mean of the first five observations, all apparently excellent, and the mean of the last six observations, all apparently excellent, differ 3·5 km./sec. for almost the same phase values, 3·97 years for the first group, and 3·77 years for the second group. The ordinates to Dr. Jones's curve for these two phase values differ less than 0·2 km./sec. It should be noted that the double amplitude of Dr. Jones's curve is only 4·1 km./sec.

In view of the several conditions just described, it seems reasonable to suppose that the probable error arising from the short-period or irregular fluctuations, designated by Dr. Jones as σ in equations (1), is very different, and larger, for the Lick observations, half of which fall in the four outlying cycles, than for the Cape observations that are concentrated in the central 1·21 cycles where the least-squares velocity curve presumably agrees best with the observed curve. These conditions can be met, *in part*, by letting σ_1 represent the value of σ for the Cape series, and σ_2 the value of σ for the Lick series. Equations (1) then become, for α Orionis,

$$\left.\begin{aligned} \sigma_C^2 + \sigma_1^2 &= 0{\cdot}53 \\ \sigma_L^2 + \sigma_2^2 &= 0{\cdot}98 \\ 2\sigma_C^2 &= 0{\cdot}55 \\ 2\sigma_L &= 0{\cdot}61 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

from which we obtain

$$\left.\begin{aligned} \sigma_C &= \text{P.E. of one Cape observation} = \pm 0{\cdot}52 \text{ km./sec.} \\ \sigma_L &= \text{P.E. ,, Lick ,,} = \pm 0{\cdot}55 \text{ ,,} \\ \sigma_1 &= \text{P.E. due to fluctuations for Cape series} = \pm 0{\cdot}51 \text{ ,,} \\ \sigma_2 &= \text{P.E. ,, ,, ,, Lick ,,} = \pm 0{\cdot}82 \text{ ,,} \end{aligned}\right\} \quad (5)$$

Dr. Jones (p. 667) had found the probable error of a difference between two consecutive Cape plates to be $\pm$ 0·74 km./sec., and between two consecutive Lick plates $\pm$ 0·78 km./sec. The squares of these numerical quantities are the 0·55 and 0·61 in the third and fourth of equations (1) and (4) above. Being confident that the probable error, $\pm$ 0·61 km./sec. in equations (3), assigned to one Lick observation, and even the revised value, $\pm$ 0·55 km./sec. in equations (5), are too large, I deduced the residuals of the Cape and Lick observations with reference to Dr. Jones's curve, and by computation found the probable error of a difference between two consecutive Cape observations to be $\pm$ 0·75 km./sec., and of a difference between two consecutive Lick observations, $\pm$ 0·76 km./sec. However, of the 64 Lick differences, only two are larger than 2·0 km./sec.: these are 2·7 and 5·0 km./sec. Now the latter is the difference between the residual for a 2-prism, 16-inch-camera observation and the residual for a 3-prism observation *made 42 days later*! Nobody knows what fluctuations in apparent velocity really occurred in the 42 days. The 5·0 difference should plainly be rejected.* The probable error of one of the remaining 63 differences between consecutive Lick observations then becomes $\pm$ 0·64 km./sec. Substituting the squares of $\pm$ 0·74 (Cape) and $\pm$ 0·64 (Lick) in the last two of equations (4), we get

$$\begin{aligned} &\text{P.E. of one Cape observation} = \pm 0{\cdot}52 \text{ km./sec.} \\ &\text{,, ,, Lick ,,} = \pm 0.45 \text{ ,,} \end{aligned}$$

It will be made clear, later in this paper, that even this small computed value, $\pm$ 0·45, is larger than the actual value of the probable error of one Lick observation of α Orionis. There appears, however, to be no occasion for carrying the numerical calculation further, since the star is clearly an unsuitable object from which to determine the probable errors of velocity measurements.

There are several reasons, given below, why the computed probable error of one Cape observation of α Orionis might have been expected to be smaller than the probable error of one Lick observation.

1. Speaking of the probable error values at which he arrived, $\pm$ 0·48 km./sec. (Cape) and $\pm$ 0·61 km./sec. (Lick), for α Orionis, Dr. Jones said, on p. 667: "The probable errors for the Cape and Lick plates (of α Orionis) are in accord with what would be anticipated from the dispersions employed." The Cape observations of this first-magnitude star were all obtained with one and the same spectrograph, con-

* Six of the 123 Cape differences were larger than 2·0 km./sec. The largest Cape difference, 3·7 km./sec., came from two 4-prism, 30-inch-camera observations made on the same night.

taining four prisms and a camera 30 inches in focal length; whereas the Lick observations, secured at Mount Hamilton and at Santiago, were made with three different 3-prism spectrographs, provided respectively with cameras having focal lengths $21\frac{1}{4}$ inches, 21 inches, and 16 inches, and with a fourth spectrograph containing two prisms and a 16-inch camera. The Cape spectrograms have the relatively large linear scale value 1 mm. = 8·4 angstroms; the Lick spectrograms have linear scale values lying between the limits 1 mm. = 10·3 angstroms and 1 mm. = 20·3 angstroms. Higher dispersion should of course make for greater accuracy in the determination of radial velocities, but it is only one of many factors to be considered. Quality of optical parts, stability of apparatus, perfection of adjustments, methods of observing and measurement, and other considerations too numerous to mention are each of importance; but efficiency and accuracy of results depend more on the correct balancing of the several factors than upon the excessive elaboration of any one of them.

2. As Dr. Jones further said: "Some of the Cape plates were measured by two or three different methods; only the measures made with the Hartmann spectrocomparator have been utilized, as the accidental errors of these measurements, which are differential in nature, are considerably smaller than those of the measurements by the other methods which were used." The Lick measurements, made in a great variety of ways, were, on the contrary, all utilized by Dr. Jones in computing the apparent probable errors of the Lick observations. Of the 83 Lick plates of α Orionis, only two secured at Mount Hamilton and three in Chile were measured with the Hartmann spectrocomparator; a few others were measured in two or more ways each; but the great majority of the spectrograms were each measured in just one of several ways—ways in many cases independent of each other. The Cape measurers, using the Hartmann spectrocomparator, were interested in determining the relative or *differential* velocities; but the Lick measurers, using a variety of measuring microscopes and methods, were interested in determining the *absolute* velocities (velocities relative to the solar system as a sidereal unit). The former method, well known and frequently used in recent years when dealing with spectroscopic binary-star orbits, must be expected, as Dr. Jones found, to give smaller "probable errors" than the latter method, but it can hardly be expected to yield absolute velocities of such dependability.

3. The Cape measures of the 141 spectrograms of α Orionis, used by Dr. Jones, were made exclusively by three Cape astronomers, Messrs. Simpson, Lunt, and Halm; and only a few observers took part, at night, in securing the spectrograms. The measures of the 83 Lick spectrograms of α Orionis were made by nineteen astronomers and assistants; and the spectrograms were secured at night by twenty-three observers.

Dr. Jones's study of the apparent velocity variations of α Scorpii was made upon identically the same plan. Of the observations used by him there had been secured—

at Mt. Hamilton,	5	with Orig. Mills spectrograph	3-pr., 16-in. camera.
,, ,,	39	,, New ,, ,,	3-pr., 21-in. ,,
in Chile,	5	,, Chile spectrograph	3-pr., 21¼ in. ,,
at the Cape,	101	,, Cape ,,	4-pr., 30-in. ,,

He plotted the Cape and Lick observations with provisional period 7·25 years. He corrected the 49 Lick observations each by − 0·26 km./sec. to make them agree, on the average, with the Cape observations. The least-squares solution resulted in a computed period of 7·35 years, and a double amplitude of 4·2 km./sec. Elliptic motion, Dr. Jones found, does not represent the observations. In this connection, the conditions described in the following paragraphs will be of interest.

The 101 Cape observations of α Scorpii extend from 1904·60 to 1912·65 ; or, if we disregard the earliest observation, which is separated by 1·5 years from the next following observation, the remaining 100 Cape observations extend from 1906·14 to 1912·75, or through 6·6 years ; that is, through *less than one period* (*7·35 years*) *of apparent orbital revolution.*

The 49 published Lick observations of α Scorpii extend from 1897·27 to 1922·37, or through 25·1 years, corresponding to 3·4 periods of apparent revolution.

Of the 49 Lick observations, 24 fall within the same time limits as the 100 Cape observations; that is, 100 + 24 = 124 observations cover only 9/10 of one cycle of variation ; and the other 1 + 25 = 26 observations are distributed over the remaining 2·5 cycles.

The orbit computed by Dr. Jones, by means of the method of least squares, seems not to represent satisfactorily several groups of Lick observations. In particular, the following recent Mount Hamilton observations (unpublished), excellent in quality, suggest that the value of the period, 7·35 years, is not right.

	Observed Velocities.	Computed Velocities.	Residuals. Obs.—Com.	Phase.
Mt. Hamilton, 1927·48	−4·0 km./sec.	−0·75 km./sec.	−3·25 km./sec.	7·24 years.
,, ,, 1928·38	−6·3 ,,	−1·50 ,,	−4·80 ,,	0·79 ,,

The average distance of these two observed velocities from Dr. Jones's velocity curve is − 4·02 km./sec. ; or, if we apply his systematic correction, − 0·26 km./sec., to these observations also, the average of the two residuals is − 4·3 km./sec.—a quantity greater than the double amplitude of the velocity curve, 4·2 km./sec., as deduced by Dr. Jones on the basis of elliptic motion.

Three Lick (New Mills spectrograph) observations of α Scorpii, made in 1917 May–August, gave residuals + 2·6, + 1·9, and + 2·4 km./sec., or mean residual + 2·3 km./sec., at mean phase 4·63 years ; and two observations made with the same instrument in 1918 July gave residuals − 1·7 and − 1·6, or mean residual − 1·65 km./sec. at mean phase 5·62 years ; and these two groups, referring to phases differing by 1·0 year, Dr. Jones combined to form one normal place !

Recent remeasurements of these five plates yield new values of the residuals : + 3·1, + 3·0, + 2·6 km./sec., and − 2·3, − 2·2 km./sec. ;

and they change the difference between the two group velocities from the original value, 3·95 km./sec., to 5·15 km./sec. The mean of these two values, 4·55 km./sec., is considerably larger than the entire double amplitude of Dr. Jones's curve, 4·2 km./sec. It should be said that the ordinates to his curve corresponding to these two phase values differ less than 0·1 km./sec. from each other. All of these facts, relating to Lick observations, suggest either that the apparent period is not 7·35 years, or that departures from elliptic motion in a phase interval of one year may be astonishingly large.

The variability of the apparent radial velocity of α Scorpii was discovered by Wright at Santiago in 1905–6. Halm, in 1909, using the observations secured by the Lick and Cape observatories in the interval 1897–1909, computed an orbit, and found a period of 5·80 years. Lunt, in 1916, concluded that the period is longer than 5·80 years. Jones has deduced a period of 7·35 years. The apparent period may be decidedly variable in length.

Dr. Jones found (p. 674), from differences of consecutive residuals, that the probable error of such a difference is ± 0·68 km./sec. for Cape plates, and ± 1·00 km./sec. for Lick plates. The squares of these numerical quantities, 0·46 and 1·00, were used by Dr. Jones as the right-hand members of the third and fourth equations of the group of four equations which defined the values of the probable errors, with results,

P.E. of one Cape observation of α Scorpii = ± 0·45 km./sec.
„ „ Lick „ „ „ = ± 0·73 „

There was no question with my colleagues at Mount Hamilton and myself that the computed value ± 0·73 km./sec. is much larger than the actual or true value, and we have sought for the explanation, with findings described in the next five paragraphs.

For the purpose of checking Dr. Jones's calculations, as far as could be done under the circumstances, I formed the successive differences of the Cape residuals with reference to his radial velocity curve, 87 in all, and found the probable error of one difference to be ± 0·81 km./sec. Seven of the 87 differences are larger than 2·00 km./sec. The largest difference is 5·9 km./sec. It concerns two observations made 19 days apart. Evidently this difference should be discarded. Dr. Jones clearly did discard it, for after I discard it I find that the probable error of one of the remaining 86 differences becomes ± 0·69 km./sec., in good agreement with Dr. Jones's value of ± 0·68 km./sec.

Dealing with the differences of the Lick residuals in the same way, 32 in all, the probable error of one difference was computed to be ± 1·02 km./sec. (Dr. Jones's computed value was ± 1·00 km./sec.). Five of the 32 Lick differences were larger than 2·00 km./sec. The largest was 4·1 km./sec. This one was the difference between the residuals for two observations made eleven months apart! Of course it should be rejected ; it should never have been brought into existence ! These two observations are the third and fourth observations com-

mented upon by me on p. 512, lines 8 and 5 from bottom, whose residuals with reference to Dr. Jones's curve are + 2·4 and − 1·7 km./sec.! The probable error of one difference for the remaining 31 residuals is ± 0·90 km./sec.

Inserting σ_1, σ_2, $0{\cdot}68^2$, and $0{\cdot}90^2$ in the group of four σ equations, we obtain for α Scorpii:

$$\begin{aligned} \sigma_C^2 + \sigma_1^2 &= 0{\cdot}50 \\ \sigma_L^2 + \sigma_2^2 &= 1{\cdot}08 \\ 2\sigma_C^2 &= 0{\cdot}46 \\ 2\sigma_L^2 &= 0{\cdot}81 \end{aligned}$$

from which are found

σ_C = P.E. of one Cape observation = ± 0·48 km./sec.
σ_L = P.E. ,, Lick ,, = ± 0·64 ,,
σ_1 = P.E. due to fluctuations for the short Cape series = ± 0·52 ,,
σ_2 = P.E. ,, ,, ,, long Lick ,, = ± 0·82 ,,

These values are decidedly different from those obtained by Dr. Jones (see p. 513); and even so, the real and correct value of the probable error of one Lick observation is very much smaller than ± 0·64 km./sec., as the next paragraph will make clear.

As already stated, the method adopted by Dr. Jones to determine the probable error of one observation, say one Cape observation of α Scorpii, consists in taking the n differences Δ between the residuals of consecutive observations, with reference to his curve representing *elliptic* orbital motion, and in computing the probable error σ of one observation, by the formula

$$\sqrt{2}\sigma = \pm 0{\cdot}6745\sqrt{\frac{\Sigma[\Delta^2]}{n-1}}.$$

This method is theoretically and practically correct when we are dealing with a star whose radial velocities are varying in strict accord with the laws of elliptic orbital motion. It is theoretically not correct if we are dealing with such stars as α Orionis and α Scorpii, whose observed radial velocities vary irregularly and in ways not comprehended. Dr. Jones was well aware of this, but he assumed that, practically, he could use the method if he applied it to successive pairs of observations for which the time interval between the two observations of a pair was so short that the *irregular* fluctuation of apparent velocity would be negligible. Now the shortness of the time intervals between successive observations is here a very critical factor. An examination of the residuals will show that irregular fluctuations of considerable magnitude seem to occur now and then within a period of a month. Dr. Jones's method cannot safely be used if the intervals between successive observations exceed a few days. The intervals between pairs of Lick observations actually employed in this manner by Dr. Jones were in many cases long: 6 weeks, 7 weeks, 8 weeks, $2\frac{2}{3}$ months, $4\frac{1}{2}$ months, 11 months! The *average time interval* for the 87 pairs of Cape observations of α Scorpii was 7·1 days; for the 32 pairs of Lick observa-

tions, 36·4 days! That explains why the value of the probable error of a Lick observation of α Scorpii, as computed by Dr. Jones, is much larger than its real value. The true value is, I doubt not, less than ± 0·40 km./sec. The computed probable error of one Lick observation, found by using only the eleven pairs of observations whose respective time intervals do not exceed seven days, is slightly smaller than ± 0·40 km./sec.

The computed values of the probable errors of the Cape and Lick observations of α Orionis are similarly affected, but in lesser degree. The *average time interval* for the 123 successive Cape pairs of observations is 5·2 days; for the 64 Lick pairs, 16·4 days; or, if we discard the one Lick observation that was plotted and used by Dr. Jones one year in error, the average for 63 Lick pairs reduces to 9·6 days. As in the case of α Scorpii, the longer interval has permitted irregular fluctuations of the star's velocity to increase the run of differences between successive residuals, to the prejudice of the calculated value of the probable error of the Lick observations.

Perhaps I may permit myself the comment that α Scorpii, by reason of its far-south position (Decl. = − 26°·3), is hardly a suitable object by which to gauge the general accuracy of observations made at Mt. Hamilton. When on the observer's meridian, its zenith distance at the Cape is 8°; at Santiago, 7°; at Mt. Hamilton, 64°. Of the 49 Lick spectrograms, 44 were secured at Mt. Hamilton and 5 at Santiago.

In the paper upon which I have been commenting, Dr. Jones gave much space to the computed probable errors of the Cape observations and the Lick observations, treated *separately*. In his other paper, on the orbits of four spectroscopic binary stars, he computed the probable error of an observation (of weight unity) of each of the four stars, but he treated the Cape and Lick observations *in combination*. He found "the probable error corresponding to unit weight (a single plate)" for the four stars to be, respectively, ± 0·92, ± 0·95, ± 0·71, and ± 0·80 km./sec. Now these four quantities are probably entirely correct for the four groups of Cape and Lick observations *in combination*, but they are very much too large for the Lick share of the combination; and because so many of my colleagues and assistants have sat up nights getting the spectrograms, and have measured the spectrograms with care and skill, it is my regrettable duty to speak on the subject.

For these four stars we need not doubt that the variations of radial velocity are the consequences of elliptic orbital motion. Accordingly, there would be no reason to distrust an analytical derivation of the probable errors of the observations. However, the formulæ for computing probable errors for one series of observations when the least-squares solution has been based upon two independent series (Cape and Lick) used in combination are rather complicated, and for this minor reason, and for a major reason which will appear later, I shall adopt a different but equally dependable method of indicating the degree of accuracy of the observations, in so far as the accidental errors are concerned. In fact, the usual method and the method here employed

depend equally and solely upon the published residuals obtained by Dr. Jones by comparing the observed values of the stellar radial velocities with the corresponding points on his computed velocity curves for the four stars.

The following table contains: first, the numbers of observations of the four spectroscopic binary stars secured by the Cape and Lick observatories respectively, with the several spectrographs * employed at the three observing stations; and, secondly, the arithmetic mean values of Dr. Jones's residuals for the Cape and Lick observatories respectively, with reference to the four velocity curves.

					β Reticuli.	τ Puppis.	λ Hydræ.	α Tucanæ.
Nos. of observations, Chile,			3-pr., 21¼-in. camera		11	15 †	..	15 ‡
,,	,,	,,	2-pr., 16-in.	,,	6 §	10	..	1
,,	,,	Mt. Ham.,	3-pr., 16-in.	,,	..	..	13	..
,,	,,	,,	3-pr., 21-in.	,,	..	..	3	..
,,	,,	Cape,	4-pr., 30-in. *	,,	5 §	7 †	9	8
,,	,,	,,	4-pr., 16-in. *	,,	12	23	14	6
Lick arithmetic mean residual, km./sec.					0·79	0·95	0·79	0·64
Cape	,,	,,	,,	,,	1·11	1·26	1·43	1·07

A glance at the residuals published by Dr. Jones for β Reticuli and for λ Hydræ suggested to me the existence of systematic differences between the Lick and the Cape observations. The suspicion was verified upon forming the *algebraic* mean values of the two sets of residuals

* The Cape 4-prism observations are divided into two groups: those which were certainly made with the 4-prism, 30-inch-camera combination, and those which seem to have been made with the 4-prism, 16-inch-camera combination. Dr. Jones does not state, in his recent paper on the orbits of these four spectroscopic binaries, whether the observations made between the years 1918 and 1925, and used in the determination of these orbits, were secured with the 4-prism, 16-inch-camera combination, or with the 4-prism, 30-inch camera; but, from other radial velocity observations made at the Cape within the period referred to, I conclude that the later observations were secured with a 4-prism, 16-inch-camera combination. Any uncertainty on this point, in its relation to the present discussion, seems not important, for the residuals attaching to the Cape observations of the four stars surely made with the 4-prism, 30-inch-camera combination—that is, those made before 1918—are larger on the average than are the residuals for the observations of the four stars secured in the period 1918–25, which were probably made with the 4-prism, 16-inch-camera combination.

† Counting two Chile 3-prism, 16-inch-camera observations, with residuals − 2·9 and − 2·2 km./sec., to which Mr. W. H. Wright, in *L.O.P.*, vol. ix., and again in vol. xvi., assigned weight ½ (because the new 16-inch camera lens he was then using was apparently defective and was soon discarded), as one observation, of weight 1, residual − 2·6 km./sec.; and omitting one Cape 4-prism, 30-inch-camera observation, residual − 5·8 km./sec., which Dr. Jones published but did not use.

‡ Counting two Chile 3-prism, 16-inch-camera observations, with residuals 0·0 and 0·0 km./sec., to which Mr. W. H. Wright, in *L.O.P.*, Vol. IX., assigned weight ½ (for the same reason as explained in the preceding footnote), as one observation, of weight 1, with residual 0·0 km./sec.

§ Not counting one Lick (Chile) 2-prism, 16-inch-camera observation rejected by Dr. Jones, residual = − 4·7 km./sec.; nor one Cape 4-prism, 30-inch-camera observation rejected by Dr. Jones, residual = + 5·2 km./sec.

for each of the four binary stars. The results thus obtained are as below :

		For β Reticuli.	For τ Puppis.	For λ Hydræ.	For α Tucanæ.
Lick algebraic mean residual,	km./sec.	−0·56	−0·17	+0·71 *	+0·03
Cape „ „ „	„	+0·60	+0·18	−0·26 *	−0·19
Cape—Lick, systematic difference,	„	+1·16	+0·35	−0·97	−0·22

In the case of α Orionis, whose apparent radial velocity was known to vary irregularly, and in an undetermined manner, Dr. Jones first of all concluded that the Lick observations, extending through 5·1 cycles of variation, were systematically 0·39 km./sec. larger than the Cape observations, extending through 1·21 cycles ; and he intended to reduce all Lick observations by this amount before applying the method of least squares to the determination of apparent orbital elements. Following identically the same logic, Dr. Jones did apply a correction of − 0·26 km./sec. to all Lick observations of α Scorpii, extending over 3·4 periods, to make them agree on the average with the Cape observations extending in effect over only 0·9 period, before computing an orbit by the method of least squares.

In the cases of β Reticuli, τ Puppis, λ Hydræ, and α Tucanæ, whose radial velocities seem to vary in strict accordance with elliptic orbital motion, the systematic differences, + 1·16, + 0·35, − 0·97, and − 0·22 km./sec., as explained above, were apparently not determined by Dr. Jones, nor were corresponding corrections made. He applied the method of least squares to the determination of the four orbits, with these systematic differences remaining in the observations.

I am assuming that these systematic differences should, first of all, have been eliminated. After subtracting from the four series of Lick residuals, as published by Dr. Jones, and likewise from the four series of Cape residuals, the quantities by which the several series varied on the average from their respective curves, as set down in the above table of algebraic means, we have the following arithmetic mean residuals ; the number of observations entering into each mean being indicated by its attached subscript :—

Arithmetic Mean Residuals.

		For β Reticuli.	For τ Puppis.	For λ Hydræ.	For α Tucanæ.	For the 4 Stars combined.
For Lick obs'ns, 3-pr., 16–21¼-in. cam.,	km./sec.	$0·63_{11}$	$0·65_{15}$	$0·46_{16}$	$0·67_{15}$	$\mathbf{0·60_{57}}$
„ Cape „ 4-pr., 30-in. cam.,	„	$1·78_{5}$	$1·43_{7}$	$1·24_{9}$	$0·94_{8}$	$\mathbf{1·30_{29}}$
„ Lick „ 2-pr., 16-in. „	„	$0·95_{6}$	$1·39_{10}$	..	$0·20_{1}$	$\mathbf{1·16_{17}}$
„ Cape „ 4-pr., 16-in. „	„	$0·61_{12}$	$1·18_{23}$	$1·56_{14}$	$1·17_{6}$	$\mathbf{1·15_{55}}$
„ Lick „ with all dispersions,	„	$0·75_{17}$	$0·94_{25}$	$0·46_{16}$	$0·64_{16}$	$\mathbf{0·73_{74}}$
„ Cape „ „ „	„	$0·95_{17}$	$1·24_{30}$	$1·44_{23}$	$1·04_{14}$	$\mathbf{1·20_{84}}$

* It is not clear why these numbers, + 0·71 and − 0·26, should differ so much from one another, for 16 residuals each + 0·71 km./sec. with reference to Dr. Jones's curve do not balance 23 residuals each − 0·26 km./sec.

Of course, the normal places and the normal equations should have been reconstructed, and a new solution by the method of least squares carried through, before determining the residuals ; but these procedures, while making some changes in the computed orbital elements, would not have altered the *relative* values of the residuals appreciably. It is chiefly for these reasons that I am not employing the precise probable-error formulæ to determine and express the accuracy of the observations.

The actual probable errors of single observations, of weight unity, are probably a little smaller, arithmetically, than the quantities in the last column of the above table; say ± **0·55** km./sec. for a Lick 3-prism, 16- to 21¼-inch-camera observation; say ± **1·10** km./sec. for a Cape 4-prism, 30-inch-camera observation; and similarly for the Lick observations made with all (four) dispersions, and for the Cape observations made with both dispersions. Much computing would be needed to arrive at more definite values.

Lick Observatory, Mt. Hamilton :
1928 *December* 21.

A Derivation of the Equations of Radiative Viscosity.

By E. A. Milne, F.R.S.

1. The phenomenon of radiative viscosity was discovered by Sir J. H. Jeans in 1926.* His first paper was followed by a second,† in which he worked out the exact equations of the transfer of momentum by radiation in a moving medium. Although Rosseland has recently ‡ suggested that radiative viscosity may be comparatively unimportant, it is a matter of interest to have the theory of radiative viscosity on as firm a base as possible as a definite chapter of mathematical physics. Jeans's method depended to a considerable extent on the use of the unsymmetrical co-ordinates of spherical trigonometry; he chose a special direction of one of his axes relative to the distribution of temperature, and then removed this restriction by a general transformation of axes. The following treatment invokes the same physical principles as were used by Jeans, but it obtains the general equations outright, and it substitutes the technique of vector and tensor analysis for the use of spherical co-ordinates, with considerably lighter algebra. The results agree with those of Jeans as regards the coefficient of radiative viscosity, but I arrive at a different numerical factor for the terms depending jointly on the temperature gradient and the velocity. These terms are given a simple physical meaning.

2. Consider a medium in motion in any manner, traversed by radiation. Let **v** be the velocity at a point **P**, of co-ordinates x, y, z. Let us temporarily take a set of axes at **P** moving with the velocity **v**. Measured in this moving frame of reference let the intensity of radia-

* *M.N.*, **86**, 328, 1926. † *M.N.*, **86**, 444, 1926. ‡ *M.N.*, **89**, 49, 1928.

tion at **P** in the direction of the unit vector ι be I, a scalar function of ι and **P**. At the neighbouring point $\mathbf{P}' = \mathbf{P} + \delta\mathbf{P}$ let the velocity be $\mathbf{v} + \delta\mathbf{v}$, and in a frame of reference moving with velocity $\mathbf{v} + \delta\mathbf{v}$ at $\mathbf{P}'$ let the direction of the ray which was along ι at **P** be $\iota' = \iota + \delta\iota$. Then according to the law of aberration we have, to the first order in $|\delta\mathbf{v}|/c$,

$$c\iota' = c\iota - [\delta\mathbf{v} - (\delta\mathbf{v} \,.\, \iota)\iota] \quad . \quad . \quad . \quad (1)$$

or

$$\delta\iota = -\frac{\delta\mathbf{v} - (\delta\mathbf{v} \,.\, \iota)\iota}{c} \quad . \quad . \quad . \quad . \quad (1a)$$

where c is the speed of light. [The notation $\mathbf{A} \,.\, \mathbf{B}$ (without necessarily brackets) denotes the scalar product of **A** and **B**.]

Suppose the pencil at **P** is contained within an element of solid angle $\Delta\omega$ round ι; at $\mathbf{P}'$ let the same pencil be contained within an element of solid angle $\Delta\omega'$ round ι'. We require to evaluate $\Delta\omega'$ in terms of $\Delta\omega$.

3. We use the symbol $\mathbf{A} \wedge \mathbf{B}$ to denote the vector product of two vectors **A** and **B**. The area of any closed plane curve is easily seen to be equal to

$$\left|\tfrac{1}{2}\int \mathbf{r} \wedge d\mathbf{r}\right|$$

taken round the contour of the area, **r** being the vector distance of a point of the contour from any fixed origin in the plane of the curve. By adding a constant vector to **r** and noting that $\int d\mathbf{r} = 0$, we see that this also represents the area if the origin is not in the plane of the curve. If now $\Delta\omega$ is an elementary solid angle surrounding a unit vector ι, we see by considering the small approximately plane area intercepted by the cone $\Delta\omega$ on the unit sphere that

$$\Delta\omega = \left|\tfrac{1}{2}\int \iota \wedge d\iota\right| \quad . \quad . \quad . \quad . \quad . \quad (2)$$

taken round the contour of the small area. Similarly

$$\Delta\omega' = \left|\tfrac{1}{2}\int \iota' \wedge d\iota'\right| \quad . \quad . \quad . \quad . \quad (3)$$

Using (1) and noticing that $\delta\mathbf{v}$ is constant round the contour, we have

$$\Delta\omega' = \tfrac{1}{2}\left|\int\left[\iota - \frac{\delta\mathbf{v} - (\delta\mathbf{v} \,.\, \iota)\iota}{c}\right] \wedge \left[d\iota + \frac{(\delta\mathbf{v} \,.\, d\iota)\iota + (\iota \,.\, \delta\mathbf{v})d\iota}{c}\right]\right|.$$

Noting that $\iota \wedge \iota = 0$, this becomes

$$\Delta\omega' = \tfrac{1}{2}\left|\int \iota \wedge d\iota - \frac{1}{c}\int \delta\mathbf{v} \wedge d\iota + \frac{2}{c}\int (\delta\mathbf{v} \,.\, \iota)\iota \wedge d\iota\right|.$$

But

$$\int d\iota = 0, \quad \text{and so} \quad \int \delta\mathbf{v} \wedge d\iota = \delta\mathbf{v} \wedge \int d\iota = 0.$$

Hence
$$\Delta\omega' = \Delta\omega\left[1+\frac{2(\iota\,.\,\delta\mathbf{v})}{c}\right] \quad . \quad . \quad . \quad (4)$$

Now let B be the intensity of black body radiation, in a frame at rest relative to the medium at **P**, for a temperature equal to the temperature of the matter at **P** measured in this frame. This means that if κ is the mass-absorption coefficient at **P** measured in the same frame, then $\kappa\rho B dv$ is the emission of radiation per unit solid angle from an element dv per second, measured in the same frame.* By the usual argument the equation of transfer is
$$(I' + dI')\Delta\omega' - I\Delta\omega = (-\kappa\rho I + \kappa\rho B)\delta s\Delta\omega \quad . \quad (5)$$
where δs is the element of arc, measured along the path of the ray, in the frame in which the medium at **P** is at rest, and $I' + dI'$ is the intensity of the pencil at $\mathbf{P} + \delta\mathbf{P}$ measured in the frame in which the medium at $\mathbf{P} + \delta\mathbf{P}$ is at rest. This is merely an application of the conservation of energy.

Using (4), (5) becomes
$$(I' + \delta I')\left(1 + 2\frac{\iota\,.\,\delta\mathbf{v}}{c}\right) - I = -\kappa\rho(I - B)\delta s \quad . \quad (6)$$

We first transform $I' + \delta I'$ to the frame moving with **P**. For this purpose we begin with the formula †
$$\frac{I'_{\nu'}\,d\nu'\Delta\omega'}{I_\nu d\nu\Delta\omega} = 1 - 2\frac{\iota\,.\,\delta\mathbf{v}}{c} \quad . \quad . \quad . \quad (7)$$

Using (4), equation (7) becomes ‡
$$\frac{I'_{\nu'}d\nu'}{I_\nu d\nu} = 1 - 4\frac{\iota\,.\,\delta\mathbf{v}}{c} \quad . \quad . \quad . \quad . \quad (8)$$

Integrating (8) over all frequencies we have
$$\frac{I'}{I} = 1 - 4\frac{\iota\,.\,\delta\mathbf{v}}{c} \quad . \quad . \quad . \quad . \quad (9)$$

Introducing this in (6) we have
$$(I + \delta I)\left(1 - 2\frac{\iota\,.\,\delta\mathbf{v}}{c}\right) - I = -\kappa\rho(I - B)\delta s\,. \quad . \quad (10)$$

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* In this sentence, and in this sentence only, dv denotes a small element of volume.

† I owe to Professor Eddington the following simple proof of this formula on quantum lines. Radiation of frequency ν incident on a small element of surface, transports energy at a rate proportional to $I_\nu d\nu\Delta\omega$. If the surface is now set in motion with velocity $\delta\mathbf{v}$, the measured energy in a frame moving with the surface will be proportional to $I_{\nu'}d\nu'\Delta\omega'$, and this must equal the original amount $I_\nu d\nu\Delta\omega$ diminished first in the ratio $1 - \iota\,.\,\delta\mathbf{v}/c$ because the quanta are overtaking the surface more slowly in this ratio, secondly in the ratio $1 - \iota\,.\,\delta\mathbf{v}/c$ because each quantum is diminished in frequency and hence in energy in this ratio. Hence formula (7). It can also be established by the usual relativity transformations.

‡ Since $d\nu'/d\nu = 1 - \iota\,.\,\delta\mathbf{v}/c$, we have $I'_{\nu'}/I_\nu = 1 - 3\iota\,.\,\delta\mathbf{v}/c$, which is in agreement with the known result that I_ν transforms like ν^3.

We next transform I and δs into their values as observed by an observer at rest. Let the suffix o denote such quantities. Then by the same argument as leads to (9) we have

$$\frac{\mathrm{I}}{\mathrm{I}_0} = 1 - 4\frac{\iota_0 \cdot \mathbf{v}}{c} \quad . \quad . \quad . \quad . \quad (9')$$

Further, by the law of aberration

$$\iota = \iota_0 - \frac{\mathbf{v} - (\mathbf{v} \cdot \iota_0)\iota_0}{c} \quad . \quad . \quad . \quad (11)$$

and we have also

$$\iota_0 \delta s_0 = \iota \delta s + \mathbf{v}\frac{\delta s}{c} \quad . \quad . \quad . \quad . \quad (12)$$

Equation (11) shows that $\iota \cdot \iota_0 = 1$ to the first order in v/c. Multiplying (12) by ι_0 scalarly we have

$$\delta s_0 = \delta s\left(1 + \frac{\mathbf{v} \cdot \iota_0}{c}\right). \quad . \quad . \quad . \quad (13)$$

Using (9′) and (13) in (10) we have now

$$\delta \mathrm{I}_0 - 2\mathrm{I}_0\frac{\iota_0 \cdot \delta \mathbf{v}}{c} + \kappa\rho\delta s_0 \mathrm{I}_0\left(1 - \frac{\mathbf{v} \cdot \iota_0}{c}\right) = \kappa\rho \mathrm{B}\left(1 + 3\frac{\mathbf{v} \cdot \iota_0}{c}\right)\delta s_0,$$

or

$$\frac{d\mathrm{I}_0}{ds_0} + \mathrm{I}_0\left[\kappa\rho\left(1 - \frac{\mathbf{v} \cdot \iota_0}{c}\right) - \frac{2}{c}\left(\iota_0 \cdot \frac{d\mathbf{v}}{ds_0}\right)\right] = \kappa\rho \mathrm{B}\left(1 + 3\frac{\mathbf{v} \cdot \iota_0}{c}\right). \quad (14)$$

We can now without loss of clarity omit the suffix o, the symbols I, δs, ι henceforth referring to quantities measured in a frame at rest; B continues to denote $(\sigma/\pi)\mathrm{T}^4$, where σ is Stefan's constant and T is the temperature measured in a local frame in which the medium at **P** is at rest.

We now write

$$d\tau = \kappa\rho ds\left[1 - \frac{\mathbf{v} \cdot \iota}{c} - \frac{2}{c\kappa\rho}\left(\iota \cdot \frac{d\mathbf{v}}{ds}\right)\right] \quad . \quad . \quad (15)$$

$$b = \frac{\mathrm{B}\left[1 + 3\dfrac{\mathbf{v} \cdot \iota}{c}\right]}{1 - \dfrac{\mathbf{v} \cdot \iota}{c} - \dfrac{2}{c\kappa\rho}\left(\iota \cdot \dfrac{d\mathbf{v}}{ds}\right)} \quad . \quad . \quad . \quad (16)$$

so that τ, b are functions of position along the path of the ray. Then

$$\frac{d\mathrm{I}}{d\tau} + \mathrm{I} = b \quad . \quad . \quad . \quad . \quad (17)$$

Equation (17) must now be integrated along the actual path of the ray. For a medium with distant boundaries the result is

$$\mathrm{I}(\iota, \mathbf{P}) = \int_{-\infty}^{\mathbf{P}} b(\mathbf{P}) e^{\tau} d\tau \quad . \quad . \quad . \quad (18)$$

To a sufficient approximation, following the usual procedure, b may be expanded in powers of τ (measured from **P**) to two terms and integrated. The result is

$$\mathrm{I}(\iota, \mathbf{P}) = b_{\mathbf{P}} - \left(\frac{db}{d\tau}\right)_{\mathbf{P}} . \quad . \quad . \quad . \quad (19)$$

We are only concerned with retaining terms involving a single differentiation with respect to τ, *i.e.* with respect to s. Thus to a sufficient accuracy

$$db = d\mathrm{B}\left[1 + 4\frac{\mathbf{v}\cdot\iota}{c}\right] + 4\mathrm{B}\frac{d\mathbf{v}\cdot\iota}{c} \quad . \quad . \quad . \quad (20)$$

whence

$$\frac{db}{d\tau} = \frac{d\mathrm{B}}{\kappa\rho ds}\left[1 + 5\frac{\mathbf{v}\cdot\iota}{c}\right] + \frac{4\mathrm{B}}{\kappa\rho c}\left(\frac{d\mathbf{v}\cdot\iota}{ds}\right) \quad . \quad . \quad (21)$$

Hence

$$\begin{aligned}\mathrm{I}(\iota, \mathbf{P}) &= \mathrm{B}\left[1 + 4\frac{\mathbf{v}\cdot\iota}{c} + \frac{2}{c\kappa\rho}\left(\iota\cdot\frac{d\mathbf{v}}{ds}\right)\right] - \frac{d\mathrm{B}}{\kappa\rho ds}\left[1 + 5\frac{\mathbf{v}\cdot\iota}{c}\right] - \frac{4\mathrm{B}}{\kappa\rho c}\left(\iota\cdot\frac{d\mathbf{v}}{ds}\right) \\ &= \mathrm{B}\left[1 + 4\frac{\mathbf{v}\cdot\iota}{c} - \frac{2}{c\kappa\rho}\left(\iota\cdot\frac{d\mathbf{v}}{ds}\right)\right] - \frac{d\mathrm{B}}{\kappa\rho ds}\left[1 + 5\frac{\mathbf{v}\cdot\iota}{c}\right]. \quad . \quad (22)\end{aligned}$$

4. From expression (22) it is only a matter of integration to obtain the flux of energy **F** (a vector) and the stress-tensor **p**. We have, in fact,

$$\mathrm{F}_\alpha = \int \mathrm{I}(\iota, \mathbf{P})\iota_\alpha d\omega \quad . \quad . \quad . \quad . \quad (23)$$

$$p_{\alpha\beta} = \frac{1}{c}\int \mathrm{I}(\iota, \mathbf{P})\iota_\alpha\iota_\beta d\omega \quad . \quad . \quad . \quad (24)$$

symbols with Greek suffixes denoting components of the corresponding Clarendon symbols. The integrals are taken over the unit sphere.

We note that

$$\frac{d\mathrm{P}_\alpha}{ds} = \iota_\beta\frac{\partial \mathrm{P}_\alpha}{\partial x_\beta} \quad . \quad . \quad . \quad . \quad (25)$$

and we require the integrals

$$\int \iota_\alpha d\omega = 0, \qquad \int \iota_\alpha\iota_\beta\iota_\gamma d\omega = 0 \; . \quad . \quad . \quad (26)$$

$$\int \iota_\alpha\iota_\beta d\omega = \frac{4\pi}{3}\mathrm{U}_{\alpha\beta} \quad . \quad . \quad . \quad . \quad (27)$$

$$\int \iota_\alpha\iota_\beta\iota_\gamma\iota_\delta d\omega = \frac{4\pi}{15}[\mathrm{U}_{\alpha\beta}\mathrm{U}_{\gamma\delta} + \mathrm{U}_{\alpha\gamma}\mathrm{U}_{\beta\delta} + \mathrm{U}_{\alpha\delta}\mathrm{U}_{\beta\gamma}] \quad . \; *(27')$$

* This result, which is easily established, is given in C. G. Darwin's paper, "The Optical Constants of Matter," *Trans. Camb. Phil. Soc.*, **23**, 153, 1924, equation (14.3).

where $U_{\alpha\beta}$ denotes the "substitution" tensor whose components are unity when $\alpha = \beta$ and zero when $\alpha \neq \beta$. [We are using only Cartesian co-ordinates in flat space, so we ignore the difference between contravariant and covariant tensors.]

We now find on inserting (22) in (23),

$$F_\alpha = \int\left[B\left\{1 + 4\frac{v_\beta \iota_\beta}{c} - \frac{2}{c\kappa\rho}\iota_\beta\iota_\gamma\frac{dv_\beta}{dx_\gamma}\right\} - \frac{1}{\kappa\rho}\iota_\beta\frac{\partial B}{\partial x_\beta}\left\{1 + 5\frac{v_\gamma\iota_\gamma}{c}\right\}\right]\iota_\alpha d\omega$$

$$= \frac{4\pi}{3}\left[\frac{4B}{c}v_\alpha - \frac{1}{\kappa\rho}\frac{\partial B}{\partial x_\alpha}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and from (24),

$$cp_{\alpha\beta} = \int\left[B\left\{1 + 4\frac{v_\gamma\iota_\gamma}{c} - \frac{2}{c\kappa\rho}\iota_\gamma\iota_\delta\frac{\partial v_\gamma}{\partial x_\delta}\right\} - \frac{1}{\kappa\rho}\iota_\gamma\frac{\partial B}{\partial x_\gamma}\left\{1 + 5\frac{v_\delta\iota_\delta}{c}\right\}\right]\iota_\alpha\iota_\beta d\omega$$

$$= \frac{4\pi}{3}BU_{\alpha\beta} - \frac{2B}{c\kappa\rho}\frac{4\pi}{15}\frac{\partial v_\gamma}{\partial x_\delta}[U_{\alpha\beta}U_{\gamma\delta} + U_{\alpha\gamma}U_{\beta\delta} + U_{\alpha\delta}U_{\beta\gamma}]$$

$$- \frac{5}{c\kappa\rho}v_\delta\frac{\partial B}{\partial x_\gamma}\frac{4\pi}{15}[U_{\alpha\beta}U_{\gamma\delta} + U_{\alpha\gamma}U_{\beta\delta} + U_{\alpha\delta}U_{\beta\gamma}]$$

$$= \frac{4\pi B}{3}U_{\alpha\beta} - \frac{2B}{c\kappa\rho}\frac{4\pi}{15}\left[\frac{\partial v_\gamma}{\partial x_\gamma}U_{\alpha\beta} + \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right]$$

$$- \frac{5}{c\kappa\rho}\frac{4\pi}{15}\left[v_\gamma\frac{\partial B}{\partial x_\gamma}U_{\alpha\beta} + v_\alpha\frac{\partial B}{\partial x_\beta} + v_\beta\frac{\partial B}{\partial x_\alpha}\right] \quad . \quad . \quad (29)$$

We observe that

$$\frac{\partial v_\gamma}{\partial x_\gamma} = \text{div } \mathbf{v}.$$

To show the presence of terms depending only on the motion and gradient-motion of the medium, we form by contraction the "mean-pressure" p defined by

$$p = \tfrac{1}{3}p_{\alpha\alpha}.$$

We find, recollecting that $\frac{1}{3}U_{\alpha\alpha} = 1$,

$$cp = \frac{4\pi B}{3} - \frac{2B}{c\kappa\rho}\cdot\frac{4\pi}{15}\cdot\frac{5}{3}\frac{\partial v_\gamma}{\partial x_\gamma} - \frac{5}{c\kappa\rho}\frac{4\pi}{15}\frac{5}{3}v_\gamma\frac{\partial B}{\partial x_\gamma} \quad . \quad (30)$$

Hence

$$c(p_{\alpha\beta} - pU_{\alpha\beta}) = -\frac{2B}{c\kappa\rho}\cdot\frac{4\pi}{15}\left[-\frac{2}{3}\frac{\partial v_\gamma}{\partial x_\gamma}U_{\alpha\beta} + \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right]$$

$$- \frac{5}{c\kappa\rho}\cdot\frac{4\pi}{15}\left[-\frac{2}{3}v_\gamma\frac{\partial B}{\partial x_\gamma}U_{\alpha\beta} + v_\alpha\frac{\partial B}{\partial x_\beta} + v_\beta\frac{\partial B}{\partial x_\alpha}\right] \quad (31)$$

5. We now put $4\pi B/c = aT^4$ (the energy-density for a medium at rest at temperature T, not the energy-density in the moving medium under discussion), T, as before stated, being the temperature measured

by a thermometer moving with the medium at any point. The Cartesian forms of (28), (30), and (31) are then

$$F_x = -\frac{1}{3}\frac{c}{\kappa\rho}\frac{\partial(aT^4)}{\partial x} + \frac{4}{3}v_x aT^4 \quad . \quad . \quad . \quad (32)$$

$$p = \frac{1}{3}aT^4 - \frac{2aT^4}{9\kappa\rho c}\operatorname{div}\mathbf{v} - \frac{5a}{9\kappa\rho c}\sum_{x,y,z} v_x\frac{\partial T^4}{\partial x} \quad . \quad . \quad (33)$$

$$p_{xx} - p = -\frac{2aT^4}{15c\kappa\rho}\left[-\frac{2}{3}\operatorname{div}\mathbf{v} + 2\frac{\partial v_x}{\partial x}\right] - \frac{a}{3c\kappa\rho}\left[-\frac{2}{3}\sum_{x,y,z} v_x\frac{\partial T^4}{\partial x} + 2v_x\frac{\partial T^4}{\partial x}\right] \quad (34)$$

$$p_{yz} = -\frac{2aT^4}{15c\kappa\rho}\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right] - \frac{a}{3c\kappa\rho}\left[v_y\frac{\partial T^4}{\partial z} + v_z\frac{\partial T^4}{\partial y}\right] \quad . \quad (35)$$

We add for completeness the expression for the energy-density E,

$$\begin{aligned} E &= \frac{1}{c}\int I(\iota, \mathbf{P})d\omega \\ &= \frac{1}{c}\int\left[B\left\{1 + \frac{4}{c}v_\alpha\iota_\alpha - \frac{2}{c\kappa\rho}\iota_\alpha\iota_\beta\frac{\partial v_\alpha}{\partial x_\beta}\right\} - \frac{1}{\kappa\rho}\iota_\alpha\frac{\partial B}{\partial x_\alpha}\left\{1 + 5\frac{v_\beta\iota_\beta}{c}\right\}\right]d\omega \\ &= \frac{1}{c}\left[4\pi B - \frac{2B}{c\kappa\rho}\frac{4\pi}{3}\frac{\partial v_\alpha}{\partial x_\alpha} - \frac{5}{c\kappa\rho}\frac{4\pi}{3}v_\alpha\frac{\partial B}{\partial x_\alpha}\right] \\ &= aT^4 - \frac{2aT^4}{3c\kappa\rho}\operatorname{div}\mathbf{v} - \frac{5a}{3c\kappa\rho}\sum_{x,y,z} v_x\frac{\partial T^4}{\partial x} \quad . \quad . \quad . \quad . \quad (36) \end{aligned}$$

so that $p = \frac{1}{3}E$. This is a general relation, which follows from the identity $\iota_\alpha\iota_\alpha = 1$.

6. Comparing (34) and (35) with the standard equations of hydrodynamics (*e.g.* Lamb, *Hydrodynamics*, 4th ed., p. 570), and noting that our p_{xx}, etc., are the negatives of the rate-of-stress components as usually defined, we see that the terms involving the gradient of the velocity are precisely those which would arise from a coefficient of viscosity

$$\mu = \frac{2aT^4}{15c\kappa\rho} \quad . \quad . \quad . \quad . \quad (37)$$

This is Jeans's result.

In addition, we see from (33) that the mean pressure p contains a term in div $\mathbf{v}$, *i.e.* proportional to the rate of expansion of the medium at the point in question. There is thus a second coefficient of viscosity (see Lamb, *loc. cit.*, p. 636), given in the usual notation of elasticity in the form

$$\lambda + \tfrac{2}{3}\mu = \frac{2aT^4}{9c\kappa\rho} \quad . \quad . \quad . \quad . \quad (38)$$

It should be noticed that this second coefficient of viscosity de-

finitely gives rise to non-vanishing terms, in general, in the equations of motion

$$\rho \frac{\mathrm{D}v_\alpha}{\mathrm{D}t} = \rho \mathrm{X}_\alpha - \frac{\partial p_{\alpha\beta}}{\partial x_\beta}.$$

7. Lastly, we have terms in (32), (33), (34), (35) depending on the temperature gradient. These terms differ from those found by Jeans in the numerical constant, but Jeans has adopted a definition of temperature which makes the energy per unit volume equal to $a\mathrm{T}^4$. This is not the local temperature (our T), which alone can have reference to the other properties of the medium at the point. To see the meaning of the terms depending on the gradient of T, let us write out the equations for the particular case when the motion is one of uniform motion at right angles to the temperature gradient. Put

$$v_x = 0, \quad v_y = v, \quad v_z = 0, \quad \frac{\partial \mathrm{T}^4}{\partial x} \neq 0, \quad \frac{\partial \mathrm{T}^4}{\partial y} = 0, \quad \frac{\partial \mathrm{T}^4}{\partial z} = 0.$$

Then

$$\mathrm{F}_x = -\frac{1}{3}\frac{ca}{\kappa\rho}\frac{\partial \mathrm{T}^4}{\partial x}, \qquad \mathrm{F}_y = \frac{4}{3}va\mathrm{T}^4, \qquad \mathrm{F}_z = 0,$$

$$p_{xx} = p_{yy} = p_{zz} = p = \tfrac{1}{3}a\mathrm{T}^4, \qquad p_{yz} = p_{zx} = 0,$$

$$p_{xy} = -\frac{a}{3c\kappa\rho}v\frac{\partial \mathrm{T}^4}{\partial x}.$$

Hence

$$p_{xy} = v\mathrm{F}_x/c^2 \quad . \quad . \quad . \quad . \quad (39)$$

The physical meaning of this is clear. Since the net flux in the x-direction is F_x, the net mass of the radiation crossing is F_x/c^2, and this, coming from material in uniform motion in the y-direction with velocity v, carries with it momentum $v\mathrm{F}_x/c^2$, which is accordingly the rate of communication of y-momentum across the x-plane, which in turn is precisely p_{xy}.

8. The physical meaning of the flux equation $\mathrm{F}_y = \frac{4}{3}v(a\mathrm{T}^4)$ can likewise be seen. This is the flux through a fixed aperture in the xz-plane, across which matter is moving with velocity v. Change to axes moving with the medium; then it is the flux of energy through an aperture moving with velocity v through a medium at rest. This is equal to the overtaking of energy-density in the stagnant medium, at the rate $v(a\mathrm{T}^4)$, together with the work done against the radiation-pressure, at the rate $vp = v(\frac{1}{3}a\mathrm{T}^4)$.

9. It may be repeated, in conclusion, that throughout this investigation T denotes the temperature of the matter recorded by a thermometer moving with the matter.

Photometry of Hydrogen and Calcium Lines in Stellar Spectra.
By Cecilia H. Payne and Emma T. R. Williams.

1. The measurement and theory of line contour have recently been brought into prominence by several investigators. The recent papers of Milne (*M.N.R.A.S.*, **89**, 3, 1928; *ibid.*, 17, 1928) contain so complete a discussion of the subject and so important a contribution to the theory that it seems to be desirable to apply his methods at once to as large a body of data as possible. The writers have been for some time engaged in the measurement of the contours of spectral lines, and a summary of the relevant data is contained in the present paper.

2. *The Material.*—The investigations by the two authors were undertaken independently, and the methods used differed somewhat. Miss Payne's material and procedure have been fully described in previous papers (*H. Repr.* 48, 1928; *H.C.* 334, 1928). Miss Williams has used substantially Dunham's method (*H.B.* 853, 1927); her material has the advantage of being quite homogeneous, having been obtained with the same telescope and prism on plates of the same emulsion, and, moreover, she has measured at least two and usually three plates of each star.

It has seemed best to keep the material in two parts in discussing the results. Miss Payne's material, however, also includes the data published by Dunham (*H.B.* 858, 1928) and by Hogg (*H.B.* 859, 1928), which have already been discussed by Miss Payne and Hogg in *Harvard Circular* 334.

3. *Theoretical Predictions.*—The specific points in Milne's discussion on which data are presented are as follows :—

(*a*) When the maximum of a line in the spectral sequence is measured for various values of the intensity ratio r, it is predicted that the greater is r, the higher is T_{max}. Near the centre of a line (high in the atmosphere) the temperature at which a given line is "widest" should be lower than for the wing of the line (deep in the atmosphere).

(*b*) Another way of expressing (*a*) is to say that the observed contours of lines should differ in shape at different temperatures.

(*c*) The actual maximum of the H and K lines of ionized calcium is of great importance, and it is expected that it will occur at a temperature of about 5000°.

(*d*) The maximum for the Balmer lines is expected at a temperature of about 11,000°.

The data bearing on each of these predictions will now be briefly presented in tabular and diagrammatic form.

4. *Maxima of Lines at Different Levels.*—The only lines at all suitable for measurement with the available dispersion are those of hydrogen and ionized calcium. As tabulated below, the lines $H\gamma$, $H\delta$, H + $H\epsilon$, and K have been measured for values of r ranging from ·96 to ·40. The tabulated quantities are half-breadths of lines, expressed in angstroms.

Table I gives, in addition to the mean of the measures for a given class, the average deviation of the mean of the measures of each star

from the mean for the class. Similarly, Table II gives the average deviation of the measures of each plate from the mean for each star.

TABLE I.

Mean Line Contour Data for Various Spectral Classes (Miss Williams's Data).
("Peculiar" stars excluded.)

Sp. Class.	No. of Stars.	No. of Plates.	Mean Value of $\lvert\lambda - \lambda_0\rvert$ for: $r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot48$.	Mean Value of r at λ_0.
				H_γ.			
B5	5	15	17·7±1·3	9·1± ·7	5·4± ·3	..	·44±·04
B8	9	27	19·9±2·0	11·2±1·5	6·9± ·9	2·5±1·0	·41±·05
B9	1	3	22·3 ..	13·9 ..	8·6 ..	4·1 ..	·34 ..
A0	22	62	29·4±3·6	16·8±1·6	11·2±1·2	5·4± ·4	·29±·02
A2	12	34	31·7±2·3	18·5±1·2	12·0±1·0	5·3± ·4	·30±·02
A3	3	9	34·0±4·1	20·0±1·6	12·3± ·9	5·1± ·3	·30±·03
A5	7	21	33·9±1·9	19·1± ·7	11·2±1·0	4·5± ·4	·33±·03
F0	8	24	23·5±3·9	13·1±2·8	7·3±1·8	3·1± ·7	·36±·04
F2	1	3	18·1 ..	8·3 ..	4·2 ..	..	·47 ..
				H_δ.			
B5	5	15	13·7±1·0	7·8± ·6	4·9± ·2	..	·40±·03
B8	9	27	16·9±1·6	9·7±1·2	6·4± ·8	2·8± ·7	·37±·06
B9	1	3	20·1 ..	11·5 ..	7·5 ..	3·8 ..	·32 ..
A0	22	62	24·5±2·8	14·5±1·4	9·7± ·8	5·1± ·3	·27±·02
A2	12	33	26·9±1·4	16·2±1·6	10·1±1·0	4·8± ·5	·28±·02
A3	3	9	28·4±2·2	16·7±1·3	10·2± ·7	4·8± ·1	·28±·02
A5	7	21	28·9± ·9	15·5± ·9	9·3± ·6	3·9± ·2	·30±·03
F0	8	24	19·0±3·5	10·8±1·7	6·6±1·5	2·6± ·6	·35±·04
F2	1	3	12·5 ..	7·3 ..	3·1 ..	..	·46 ..

TABLE II.

Mean Line Contour Data for Certain Stars (Miss Williams).

Sp. Class.	Star.	H lines.	No. of Plates.	Mean Value of $\lvert\lambda - \lambda_0\rvert$ for: $r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot48$.	Mean Value of r at λ_0.
					H_δ.			
A0	o Her	Narrow	3	17·6± ·6	10·7± ·3	6·8± ·2	3·8±·2	·31±·03
	β UMa	Average	3	24·1±1·5	14·6±1·4	10·6± ·6	5·3±·3	·26±·01
	α Lac	Broad	3	29·2± ·9	18·0±1·0	11·6±1·3	5·9±·8	·24±·03
A5	α Cep	Narrow	3	26·4±2·9	13·8±1·0	8·1± ·0	3·6±·2	·34±·02
	ι UMa	Average	3	28·8± ·8	15·2±1·4	9·2± ·5	3·8±·2	·31±·00
	δ Cas	Broad	3	30·2±1·7	17·8± ·7	10·8± ·6	4·8±·5	·26±·03

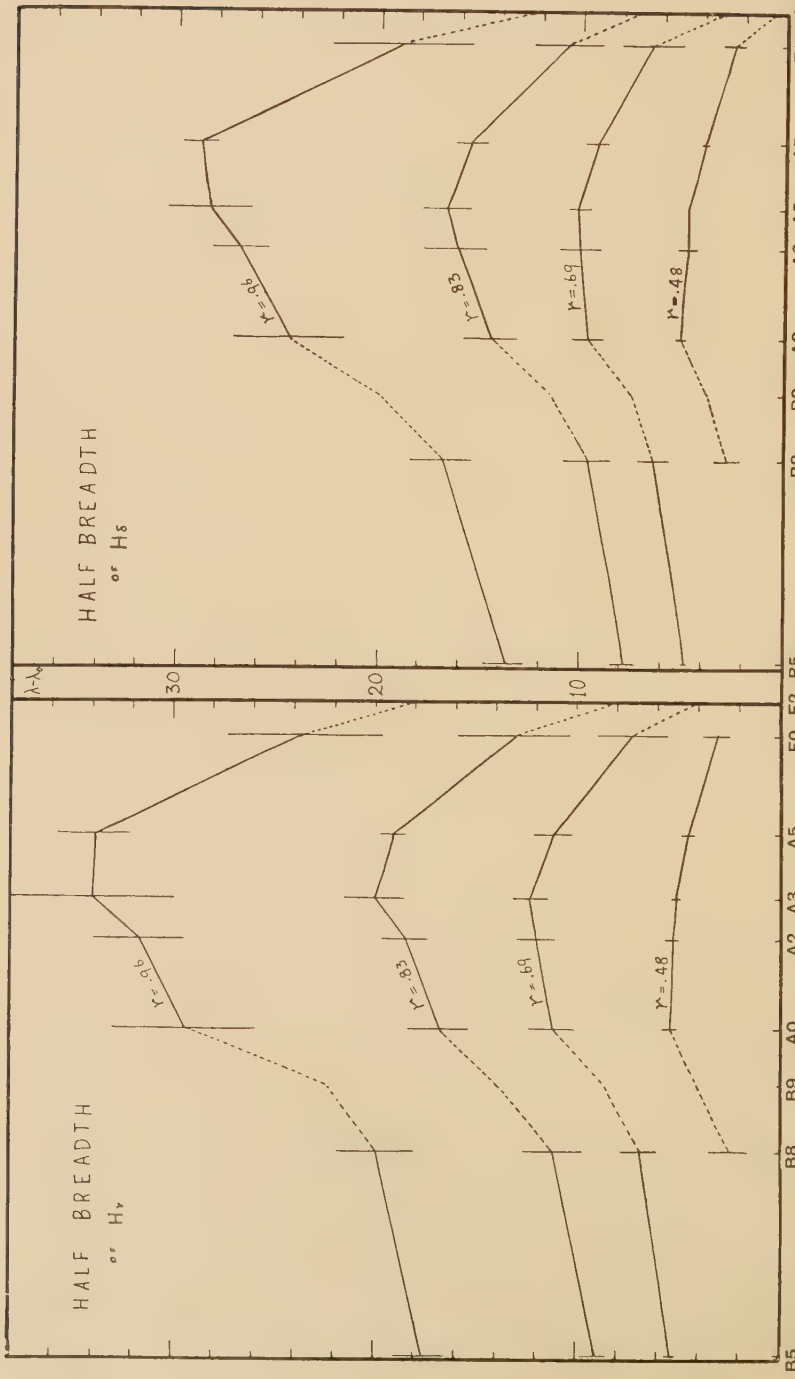

FIG. 1.—Relation of Half-breadths of Hydrogen Lines to Spectral Class (Miss Williams). Ordinates are half-breadths; abscissæ are spectral classes, spaced according to approximate temperatures.

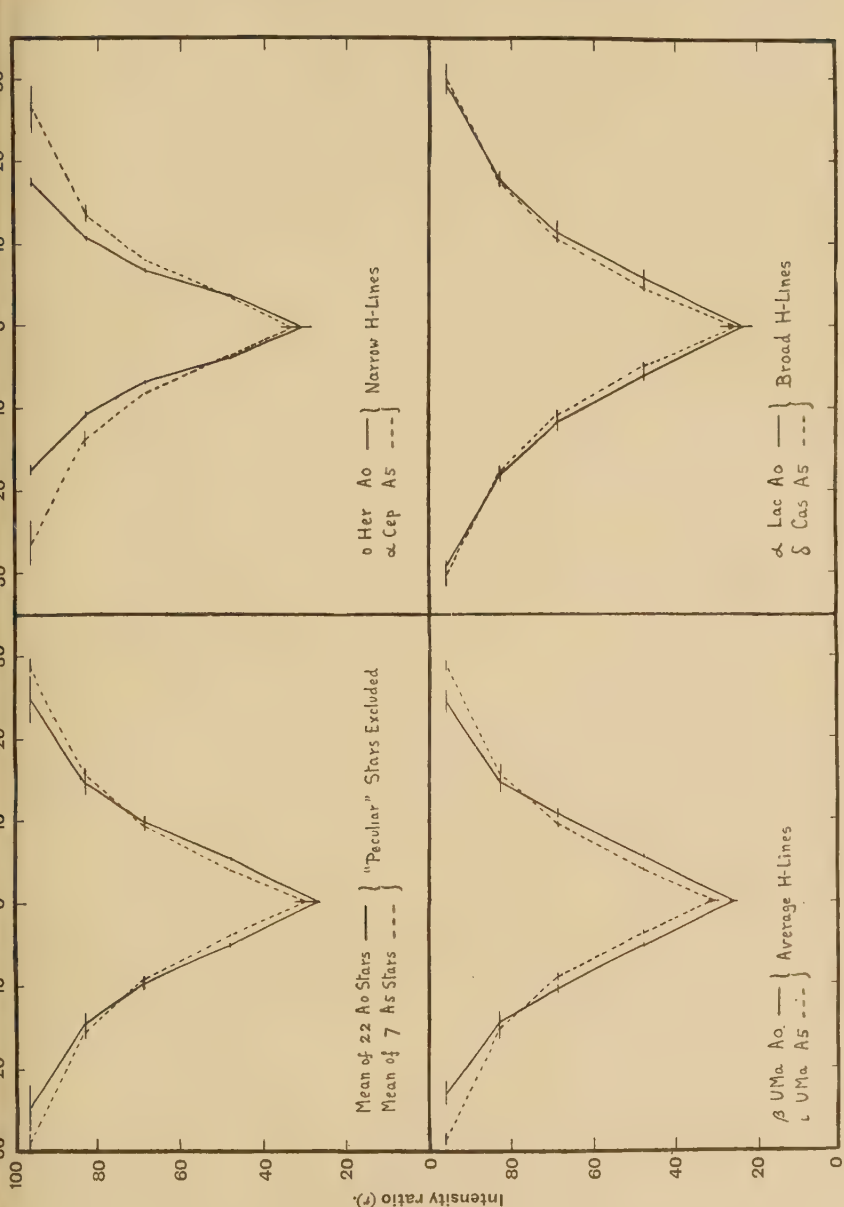
Intensity ratio (r).
Mean of 22 Ao Stars ——
Mean of 7 As Stars - - -
"Peculiar" Stars Excluded
β UMa Ao. ——
ι UMa As - - -
Average H-Lines
o Her Ao ——
α Cep As - - -
Narrow H-Lines
α Lac Ao ——
δ Cas As - - -
Broad H-Lines

FIG. 2.—Comparison of Mean Contours and Individual Contours of *Hδ* for Classes Ao and A5 (Miss Williams). Abscissæ are angstroms; ordinates are values of *r*.

TABLE III.

Mean Values of $|\lambda - \lambda_0|$ *for Various Spectral Classes (Miss Payne's Data).*

Spectral Class.	Line.	No. Stars.	$r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot58$.	$r=\cdot48$.
O	$H\gamma$	3	7·8	3·7	2·2	1·3	..
B2		1	14·0	6·0	..	..	..
B5		4	18·4	8·0	4·2	3·0	2·6
A2*		3	33·2	19·2	12·8	9·8	7·2
A5		3	33·7	14·4	8·4	5·3	4·2
F0		1	21·5	12·0	7·5	4·8	3·2
F2		1	17·2	8·6	3·7	2·4	1·6
F5		2	12·9	6·8	3·8	2·4	..
K5		1	2·0	..	..	..	..
O	$H\delta$	3	4·4	2·3	1·1	..	..
B2		1	10·2	2·7	0·6	..	..
B5		4	15·9	7·2	3·8	3·0	2·0
A2*		3	36·6	22·4	14·7	10·6	7·8
A5		4	34·4	15·3	8·8	5·8	3·8
F0		2	19·5	10·3	5·7	3·5	2·8
F2		1	18·5	10·2	5·4	3·0	1·1
F5		2	12·9	6·5	3·6	2·1	0·6
K5		1	3·0	..	..	..	..

* Three stars with abnormally strong hydrogen lines.

TABLE IV.

Mean Values of $|\lambda - \lambda_0|$ *for Class A (Dunham's Data).*

Spectral Class.	Line.	No. Stars.	$r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot58$.	$r=\cdot48$.
A0	$H\gamma$	11	27·0	15·1	9·1	6·5	4·6
A2		2	28·2	14·4	9·2	6·2	4·4
A5		1	30·6	14·4	8·2	5·0	2·6
A0	$H\delta$	11	25·5	12·9	9·4	6·9	5·1
A2		2	26·8	13·6	9·2	5·8	3·4
A5		1	33·0	14·0	7·8	5·9	3·6
A0	H + $H\varepsilon$	11	22·4	13·3	9·4	7·0	5·6
A2		2	24·4	14·1	9·2	6·2	4·2
A5		1	32·0	15·1	8·2	5·4	3·2

It would seem from the diagrams 1, 3, and 4 that the maximum of the lines of hydrogen differs for different levels; but the difference is in the sense that the value of T_{max} is lower, not higher as predicted, for large values of r than for small ones. For $r = \cdot96$ the maximum

apparently occurs at A3–A5; for $r = \cdot48$ it is at Ao.* This effect is shown independently by the data of Miss Williams and Miss Payne.

TABLE V.

Mean Values of $|\lambda - \lambda_0|$ *for Various Spectral Classes (Miss Payne's Data).*

Spectral Class.	Line.	No. Stars.	$r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot58$.	$r=\cdot48$.
O	H + *Hε*	4	4·2	1·6	0·7	0·3	..
B2		1	16·2	12·7	3·2	1·2	..
B5		5	21·0	10·8	6·3	3·8	2·8
A2*		3	35·8	20·7	14·5	13·2	8·5
A5		4	33·4	17·6	9·8	7·7	5·7
A7		1	18·6	10·3	6·3	4·6	4·0
A8		1	20·5	11·3	7·2	5·4	4·0
F5		2	11·2	8·6	6·7	5·6	4·7
F8		2	13·2	8·6	6·9	5·8	4·8
Go		2	13·4	9·6	7·4	6·4	5·3
G5		3	11·7	9·6	8·1	6·6	5·1
Ko		6	25·4	13·3	9·1	7·1	5·9
K2		3	27·3	15·3	9·6	7·0	5·3
K2 (Hogg)		2	25·8	12·5	9·2	7·4	5·8
Mo	,,	6	19·0	10·2	6·3	4·8	3·6
M1	,,	1	20·5	9·5	4·5	3·3	2·5
M2	,,	1	17·8	7·5	5·0	3·8	2·8
M3	,,	2	16·1	7·9	5·2	3·6	2·5
M5	,,	2	11·9	6·6	4·1	3·2	2·4

* Three stars with abnormally strong hydrogen lines.

TABLE VI.

Mean Values of $|\lambda - \lambda_0|$ *for Various Spectral Classes (Miss Payne's Data).*

Spectral Class.	Line.	No. Stars.	$r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot58$.	$r=\cdot48$.	$r=\cdot40$.
Ao	K	1	2·5	0·8	0·5	..	..	..
A2		4	5·9	2·4	0·5	..	..	..
A3		1	6·7	4·0	2·5	1·8	1·5	1·3
A5		1	8·5	5·5	3·8	3·0	2·5	2·0
A7		1	9·3	5·6	4·4	3·8	3·2	2·8
A8		1	11·5	7·0	5·3	4·2	3·1	1·7

* It should be noted that an explanation for this contradiction of the theory would be forthcoming if it could be shown that the line representing continuous background on the microphotometer tracings of stars of classes A3 and A5 has been drawn systematically too high by at least 0·06 magnitudes (*cf.* fig. 8). This seems improbable, particularly so inasmuch as the effect is shown by both sets of data.

TABLE VI.—*continued.*

Spectral Class.	Line.	No. Stars.	$r=\cdot96$.	$r=\cdot83$.	$r=\cdot69$.	$r=\cdot58$.	$r=\cdot48$.	$r=\cdot40$.
F0	K	4	13·2	7·5	5·5	4·3	3·8	3·3
F2		2	16·0	10·0	7·5	6·5	6·0	5·0
F8		3	22·2	12·8	8·6	6·5	4·8	3·9
G0		1	18·0	11·0	8·0	6·0	5·0	4·0
K0		5	23·4	16·8	12·4	10·9	8·2	7·2
K2		3	30·6	20·6	15·0	11·0	8·6	7·0
K2 (Hogg)		2	28·0	16·1	11·8	9·4	7·1	5·7
K5	,,	1	19·4	14·2	9·7	6·4	4·2	3·0
M0	,,	6	22·2	15·8	10·9	8·2	6·2	4·8
M1	,,	1	27·2	18·0	10·0	6·8	4·8	3·2
M2	,,	1	22·3	15·6	10·6	7·9	6·1	5·0
M3	,,	2	22·2	10·8	7·0	5·0	4·0	3·2
M5	,,	2	23·2	15·8	10·2	7·9	5·1	3·8

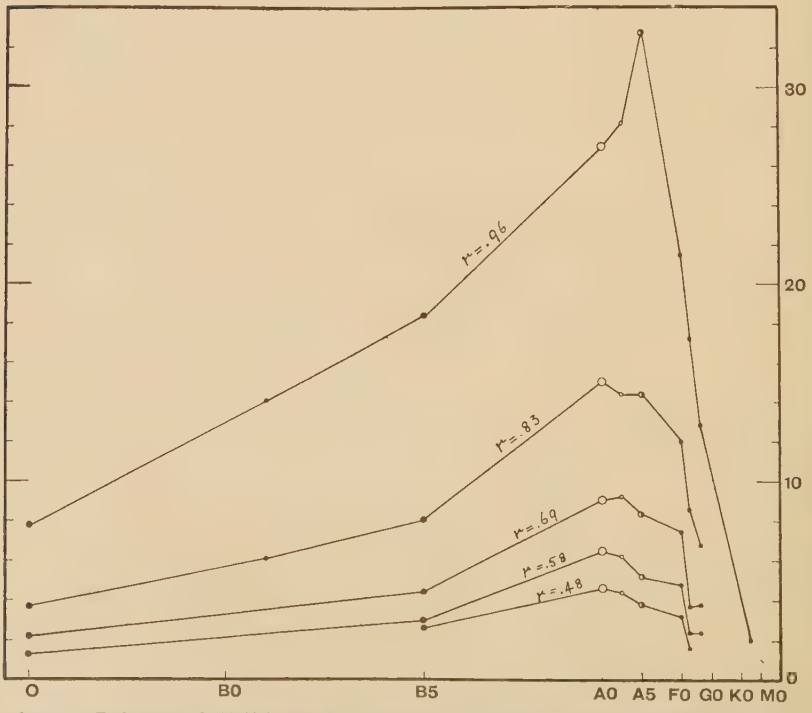

FIG. 3.—Relation of Half-breadth (ordinates) to Spectral Class (abscissæ) for $H\gamma$ from Miss Payne's data. Small, medium, and large dots represent the weights of points, according to the numbers of stars contributing to the means. Values plotted with open circles are data by Dunham.

For the lines of calcium the case is not so clear, though there is apparently a very slight effect in the same sense as that observed for hydrogen. Results for the H line are to be considered of greater weight than for the K line, because in the spectral classes at which the maximum occurs the background for the K line is very difficult to trace. It should be noted, however, that the results of Miss Payne and Hogg, for which the backgrounds were drawn independently, are closely

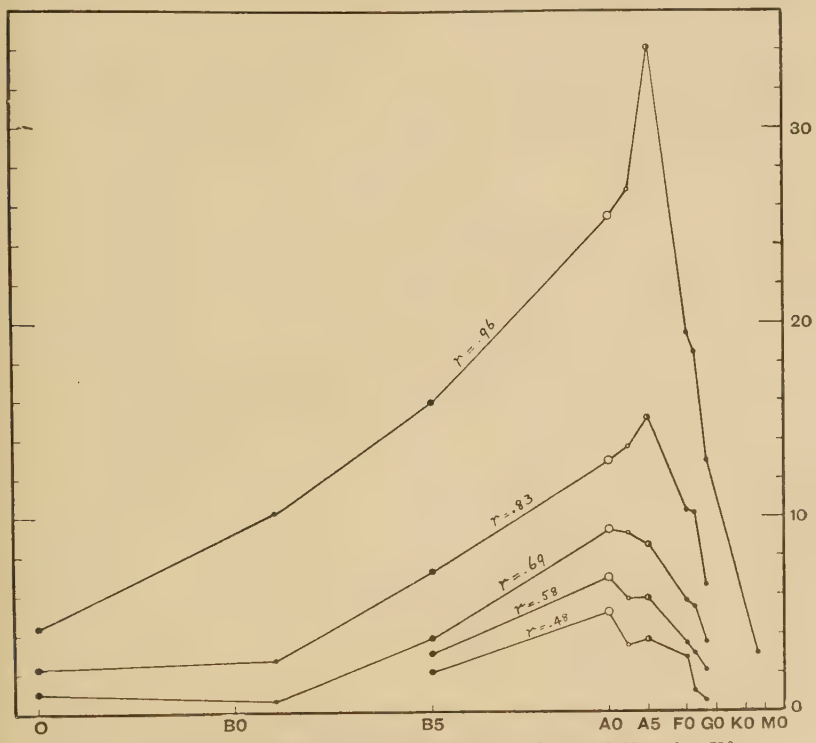

FIG. 4.—Relation of Half-breadth (ordinate) to Spectral Class for *Hδ* (Miss Payne's data).

accordant for Class K2. The difference between the two results is of the order of the actual difference that is found for stars of similar spectral class.

5. *Shapes of the Observed Contours.*—Another way of expressing the result embodied in the preceding tables is to say that the higher the temperature the narrower are the extreme wings in comparison with the rest of the line. Individual contours show this effect well, as do the means contained in the first tables. The contours of the various lines examined are reproduced in figs. 2 and 7. Fig. 8 shows the actual microphotometer tracings made by Miss Williams for a star of Class A0 and for one of Class A5. Fig. 2 and Table II compare individual stars

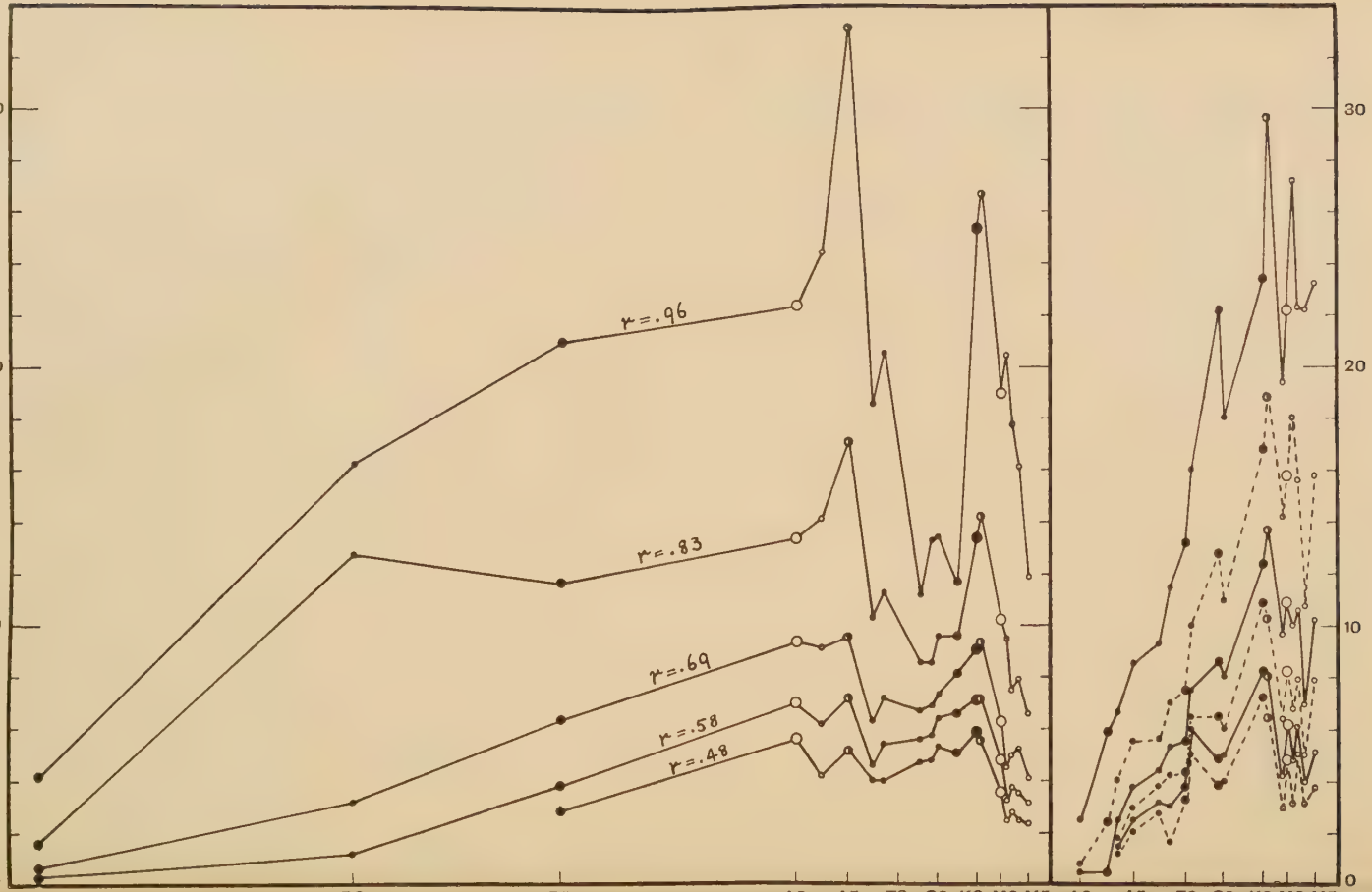
30
20
10
0
r=.96
r=.83
r=.69
r=.58
r=.48
O
B2
B5
A0
A5
F0
G0
K0
M0
M5
A0
A5
F0
G0
K0
M0
M5
30
20
10
0

Fig. 5

of Ao and A5. Thus, among the non-peculiar Ao stars examined, o Her has the narrowest H lines; it is compared with α Cep, the corresponding star for Class A5. The broader wings and shallower centres of the lines in the A5 stars are apparent in all the comparisons. This would be even more striking if two stars with the same central intensity

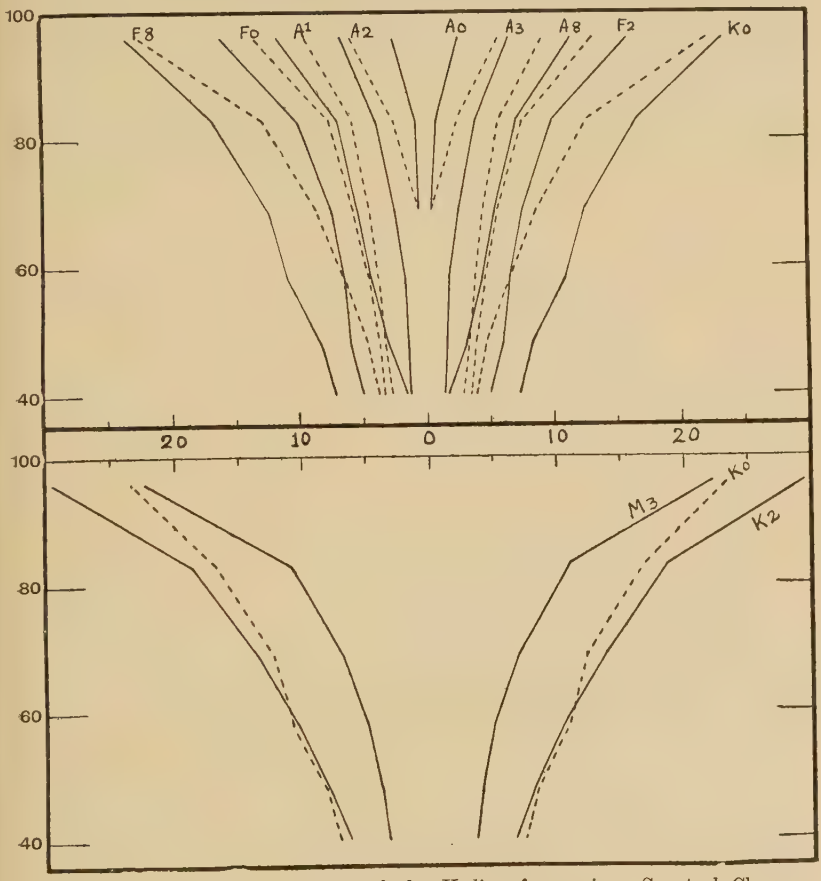

Fig. 7.—Comparison of Contours of the K line for various Spectral Classes. Ordinates are values of r; abscissæ are angstroms.

were compared, *i.e.* o Her compared with ι UMa and β UMa with δ Cas. Fig. 7 contains similar results from Miss Payne's data for various spectral classes. The increasing wing with low temperature may be traced throughout the series.

6. *The Maximum for Ionized Calcium.*—Both the calcium lines definitely have maxima within Class K, probably at Class K2 for the wings. This is in accordance with estimates of line intensity and with measures discussed on the basis of the Unsöld fit (*H.R.* 48, 1928),

which showed an evident maximum at the same spectral class. The suggested maximum at or above 5000° is therefore not substantiated. Probably the temperature for Class Ko is not greater than 4200°; temperatures based on radiation measurements cease to be of much weight in classes for which band absorption is on the verge of coming into prominence, and it is likely that the temperatures for the K and M stars are even lower than is supposed at present. There is, indeed, an indication that the temperature scale is too high almost all along the spectral sequence.

7. *The Maximum for Hydrogen.*—The wings of the hydrogen lines have a maximum at Class A3–A5, and the more central regions of the

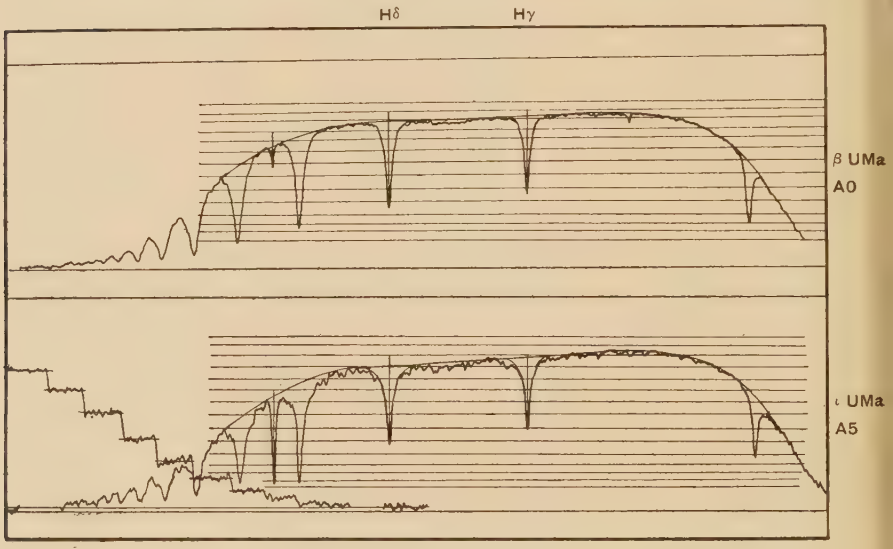

FIG. 8.—The parallel lines are ruled at intervals of 0·2 magnitudes: the "steps" at the left show the method of calibration.

lines at Class Ao. The latter class may be at a temperature of 11,000°, but is probably not higher, and Class A5 is certainly much lower. The classification of the A stars is at present so unphysical that it is not possible to be sure of the temperature of a given star when its spectral class has been assigned in the Draper System, but there is no doubt that the mean of a number of stars of Class A5 is at a lower temperature than the mean for Class Ao.

8. *Data for Supergiant Stars.*—The powerful methods opened up by the idea of considering an isolated level instead of an integrated line should permit the comparison of the atmospheres of stars of different luminosities. The next table contains a concise comparison of the contours of lines for normal and supergiant stars. The data for the latter are not yet complete enough for the determination of maximum at different levels, but by analogy with sections (4) and (5) above

the same effect could be detected from the change in the shapes of individual contours. There is another reason for not examining the data for supergiants for shift of maximum. The analysis of any set of data in that manner presupposes that the atmospheres of the stars concerned are homologous. It is evident that the atmospheres of supergiants are not homologous with those of giants, and, further, unpublished material indicates clearly that individual supergiants differ greatly in atmospheric structure. To a lesser extent the same is undoubtedly true of the normal giants, and this deprives the observed maxima of some of their significance. Probably, however, the atmospheres of the giant stars are homologous enough for the present investigation to be justified.

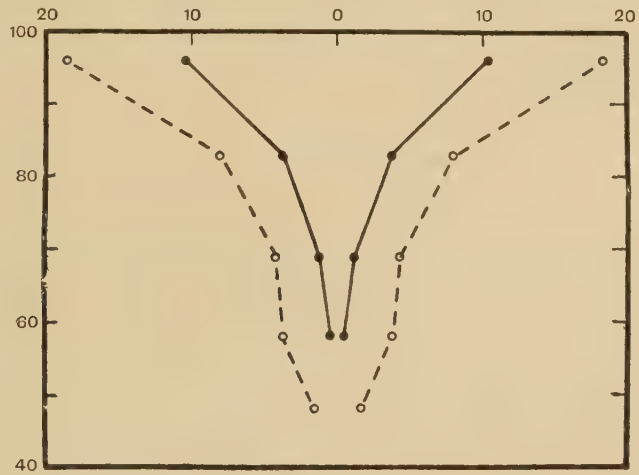

FIG. 9.—Comparison of Contours for Class B5 (below) and Class cB5. Ordinates are values of r; abscissæ are angstroms.

TABLE VII.

Comparison of Contours for Normal and Supergiant Stars (Miss Payne's Data).

Spectral Class.	Line.	No. of Stars.	r=·96.	r=·83.	r=·69.	r=·58.	r=·48.	r=·40.
cB5	*Hγ*	6	10·3	3·8	1·1	0·6	..	..
B5		4	18·4	8·0	4·2	3·8	1·6	..
cB5	*Hδ*	6	8·7	4·3	1·8	0·3	..	..
B5		4	15·9	7·2	3·8	3·0	2·0	..
cB5	H + *Hε*	6	11·0	4·8	2·2	0·8	..	..
B5		5	21·0	10·8	6·3	3·8	2·8	..
cB5	K	1	5·8	2·6	..	..	..	..
B5		1	0·4	..	..	..	..	..

9. *Conclusion.*—It does not at present seem to be profitable to go further in the comparison of theory and observation. The data contained in the present paper do not appear to be in accordance with the arrangement of a stellar atmosphere suggested by Milne, and therefore we have not proceeded to evaluate absolute abundances according to the method that he outlines.

The results of the investigation are definite in disagreeing with the predictions referred to in sections 3*c* and 3*d*; however, the predicted shift of maximum referred to in sections 3*a* and 3*b* is contradicted definitely by the hydrogen lines only. Since the hydrogen lines are anomalous in other respects, perhaps they may be in this one too.

The results lay especial stress on the value of considering the changes of a line at various levels instead of regarding it as arising, for the purposes of the investigation, at one part of the star's atmosphere.

Harvard College Observatory.

The Physical Conditions in New Stars. By the late S. R. Pike, M.A.

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Although the true cause underlying the sudden flare up of new stars remains completely obscure, the idea seems to be gaining ground that the effects are due to some sort of explosion taking place in a single star. It has been shown * that the simplest explanation of the mixed bright and dark line spectrum is provided by the hypothesis of one or more expanding shells of gas. These expanding shells are presumably the chromosphere and reversing layer of the star, and the question therefore naturally arises whether the photosphere of the star also moves outwards to an appreciable extent. The following simple argument shows that this must actually occur.

When a star increases in brightness there are just two ways in which it can do so: it can either raise its surface temperature or it can increase its radius. In general, if m_0, m are the initial and final magnitudes,

$$m_0 - m = 5 \log_{10} \frac{RT^2}{R_0 T_0^2} \quad . \quad . \quad . \quad (1)$$

where R_0, T_0, and R, T are the initial and final radii and temperatures respectively.

In this equation m_0 and m are the *bolometric* magnitudes (*i.e.* the magnitudes reckoned for radiation of all wave-lengths), and to obtain

* Evershed, *Monthly Notices*, **79**, 483, 1919; Adams, *Jour. R.A.S. Can.*, **13**, 75, 1919.

these from the visual magnitudes (*i.e.* the values actually determined) a correction has to be applied to the latter. We have

$$m_{bol} = m_{vis} - \Delta m$$

where Δm is always positive, and is a function of the surface temperature.

Values of Δm have been calculated by Eddington * for temperatures between 2500° A. and 12,000° A.; and he also gives values up to 20,000° A. derived by extrapolation of an empirical formula due to Hertzsprung. It seems possible, however, that the surface temperature of a new star may be very much higher than this; and the first step therefore is to calculate Δm directly for a more extended range. The procedure is as follows :—

We have

$$\left.\begin{aligned} \Delta m &= \tfrac{5}{2}\log_{10} p \\ \text{where} \quad p &= \frac{\int_0^\infty p(\lambda)\mathrm{E}(\lambda, \mathrm{T})d\lambda}{\int_0^\infty \mathrm{E}(\lambda, \mathrm{T})d\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

Here $\mathrm{E}(\lambda, \mathrm{T})$ is Planck's radiation function, and $p(\lambda)$ is the luminous efficiency of light of wave-length λ.

This latter quantity has been measured by Nutting,† who gives values for 31 different wave-lengths between 4000 A. and 7000 A. The numerator in the expression for p can accordingly be evaluated by numerical integration; and Weddle's rule, which makes use of seven ordinates, and so can be applied five times, is very convenient for the purpose.

The process was carried out for various temperatures up to 60,000° A., and it was found that on plotting $\log \Delta m$ against $\log \mathrm{T}$ that a smooth curve was obtained which could be extrapolated with some confidence as far as 100,000° A., or even a little further. The values of Δm are given in the adjoined table, those found by extrapolation being in brackets.

Table I.

T	12,000	20,000	25,000	30,000	35,000	40,000
Δm	0·54	1·60	2·16	2·63	3·10	3·45
T	50,000	60,000	70,000	80,000	100,000	120,000
Δm	4·10	4·64	(5·2)	(5·6)	(6·35)	(6·7)

We can now apply equation (1) to find the changes in radius and temperature of a nova, provided we know the initial conditions.

* *Internal Constitution of the Stars*, p. 138, 1926. See also Rabe, *Astr. Nach.*, **225**, 223, 1925.

† *Phil. Mag.*, **29**, 304, 1915.

Unfortunately there is as yet only one star for which all the necessary data are available, viz. Nova Aquilæ No. 3 (1918). For this star *

$m_0 = 10{\cdot}5$, $m = -1{\cdot}4$ (visual magnitudes), $D = 360$ parsecs,

and before the outbreak the spectrum was of type A, so that we may take

$$T_0 = 10{,}000^\circ \text{ A.}$$

and find at once

$$R_0 = 6{\cdot}3 \times 10^5 \text{ km.}$$

Since the absolute magnitude before the outbreak was 2·7, the star appears to have been a normal A-type dwarf, with a mass $1\frac{1}{2}$ to 2 times that of the sun.

Between June 7 and 10 it rose in brightness by 11·9 magnitudes, *i.e.* its emission of light increased in the ratio $5{\cdot}75 \times 10^4$ to 1 ; and since until just before maximum the spectrum was continuous with only a few faint absorption lines, this increase in brightness must have been simply due to changes in radius and temperature.

By assuming various values for the temperature at maximum, we can find at once the corresponding values of the radius from equation (1), after applying the appropriate correction Δm to the visual magnitude. The maximum value of R/R_0 occurs, of course, when there is no change in T, and is at once found to be

$$\frac{R}{R_0} = 240,$$

whilst for other assumed temperatures we have the values given in the table.

TABLE II.

T	12,000	20,000	25,000	30,000	40,000	60,000	100,000
$\frac{R}{R_0}$	190	110	91	79	65	49	39

It will be seen at once from these figures that it is not possible to account for the change in brightness by means of a rise of temperature alone, unless we assume fantastically high values for such a rise. It seems certain that there must be a large change in radius as well ; and, moreover, we can make a pretty close estimate of the actual value of this change. For it seems likely *a priori* that the temperature at maximum will be somewhere between 25,000° and 60,000°, and the permissible values of R/R_0 are then confined to the narrow limits

$$\frac{R}{R_0} > 50 \quad \text{and} \quad < 90.$$

Since $R_0 = 6{\cdot}3 \times 10^5$ km., we may take it that R must be of the order 5×10^7 km.

Now the rise in brightness takes place in about 3 days, or $2{\cdot}5 \times 10^5$

* Hubble and Duncan, *Astrophys. Jour.*, **66**, 59, 1927.

secs. say, so that if the change in radius is carried out with a uniform velocity v, then v must be of the order 200 km./sec.

It is, of course, the photosphere of the star which expands with this velocity; the actual outward velocity of the gas may be quite different, as the following simplified example will show.

Consider a cloud of gas which is being emitted with constant velocity V and constant initial density ρ_0 from the surface of a sphere r_0. At any time t let the outer boundary be at r_1, where the density is now ρ_1, and let the photosphere be at (r, ρ). Now the photosphere is at a definite "optical depth" τ, so that

$$\int_{r_1}^{r} k\rho dr = \tau$$

where k = mass-absorption coefficient.

Now

$$\rho r^2 = \rho_1 r_1^2 = \rho_0 r_0^2 = \mathrm{K} \text{ say},$$

$$\therefore \quad \tau = k\mathrm{K}\left(\frac{1}{r_1} - \frac{1}{r}\right),$$

and hence

$$\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{d}{dt}\left(\frac{1}{r_1}\right).$$

Further,

$$\frac{dr_1}{dt} = \mathrm{V} \quad \text{and} \quad \frac{dr}{dt} = v,$$

the apparent velocity of expansion of the photosphere,

$$\therefore \quad \mathrm{V} = \frac{r_1^2}{r^2} v.$$

Thus in all cases $\mathrm{V} > v$, and may easily be as much as 5 or 10 times greater.

In general, whatever be the actual distribution of density in the shell, so long as it increases inwards we shall have as a rule $\mathrm{V} > v$, with $\mathrm{V} = v$ as the limiting case.

Thus we are led to the conclusion that during the early stages of Nova Aquilæ there must have been an outrush of gas taking place with a velocity not less than 200 km./sec., and probably a good deal faster.* It has already been mentioned that the hypothesis of an expanding shell of gas provides the most natural explanation of the later developments in the spectrum of a nova; but it does not seem to have been previously recognized that the mere increase in brightness alone requires a similar hypothesis to explain it.†

The expansion cannot, of course, continue indefinitely, for presently the star begins to fade; but at the same time the gas which is actually moving outwards at the time of maximum must continue to do so, as its velocity exceeds the parabolic value for the star. An inward

* Compare *Monthly Notices*, **79**, 279, 1919.

† See, however, Lunt, *Monthly Notices*, **86**, 498, 1926.

contraction of the photosphere is, however, quite possible, even though the gases themselves are moving outwards; we have only to suppose that the velocity and density of the shells that are being shot off from the inner core begin ultimately to diminish. There will then be a sort of "clearing" of the atmosphere; we shall be able to see further and further into the gas as time goes on, and the photosphere will actually appear to be contracting.

Our present line of argument will not serve to carry us any further than this. It depends essentially upon the assumption that the star radiates like a black body, and after the maximum the light becomes increasingly concentrated in the bright line spectrum. If measurements were available of the light in the continuous spectrum, they would serve our purpose; but none appear to have been made. There is no doubt, however, that the continuous spectrum fades very much more rapidly than the light of the star as a whole.

A more hopeful means of studying the condition of the star in its later stages is provided by the theory of thermal ionization; and as it happens, the theory as applied to an expanding shell of gas assumes a remarkably simple form. Since the substances composing the shell all appear to travel outwards with the same velocity, the composition of the shell should remain unchanged; and any changes in the relative intensities of the various bands of the spectrum should be due simply to changes in ionization.*

The degree of ionization is controlled by the radiation from the central star, but the gas is not, of course, in thermodynamical equilibrium with it. The radiation is "evenly diluted," and, following Eddington,† we may calculate the ionization by assuming a fictitious electron density σ_e, and a temperature T, where $\sigma_e = \delta\rho'$, δ being the dilution factor of the radiation, ρ' the actual electron density, and T the black body temperature of the star.

Let

R_0, R = initial and present radii of the star.
d = present radius of shell.
ρ_0, ρ = initial and present densities of shell.
ρ_0', ρ' = initial and present *electron* densities of shell.

Then

$$\delta = \frac{d^2}{R^2} \quad \text{and} \quad \rho d^2 = \rho_0 R_0^2.$$

Let

$$\rho_0 = k_0\rho_0', \qquad \rho = k\rho'.$$

Then

$$k\rho' d^2 = k_0\rho_0' R_0^2,$$

and so we get

$$\sigma_e = \rho'\delta = \frac{k_0}{k} \cdot \left(\frac{R_0}{R}\right)^2 \rho_0' \quad . \quad . \quad . \quad . \quad (3)$$

* Apart from the possible effects of "metastability," *v.* Eddington, *Monthly Notices*, **88**, 134, 1927.

† *Proc. Roy. Soc.*, **111**, 424, 1926 (Bakerian Lecture).

To a first approximation we may take $k_0/k = 1$, *i.e.* we assume that there is a plentiful supply of free electrons present, provided by elements of low ionization potential, so that the electron density will be more or less independent of the state of ionization of the more refractory substances present; these alone are likely to vary. From the figures given later it will be seen that the approximation must be very closely obeyed.

From equation (3) we therefore have the following simple rule: The degree of ionization of the gas in the shell is the same as that in a gas in which the electron density is $\left(\frac{R_0}{R}\right)^2$ times the initial electron density of the shell.

After maximum, the value of R must decrease, as we have seen; this raises the value of the fictitious electron density, and so tends to retard ionization. Hence unless the temperature continues to rise, there must be a progressive decrease in the degree of ionization of the expanding shell.

The significant changes in the spectrum of Nova Aquilæ may be summarised as follows:—*

1918 July–Sept.	*He*$^+$ 4686 strong *He* 4471 present	*H* Balmer lines strong, weakening.	Nebulium lines (O^{++}) strong, strengthening.
1919	*He*$^+$ 4686 strong	*H* Balmer lines weak.	Nebulium lines very strong.
1926–27	*He*$^+$ 4686 strong	*H* Balmer lines strong.	Nebulium lines very faint.

The evidence points quite definitely to an increase in the degree of ionization, *i.e.* in the temperature, for a long time after maximum. The strengthening of N_1N_2 and the weakening of the Balmer lines between 1918 and 1919 may only be a relative effect, due to the metastable origin of the nebulium lines;† but the disappearance of λ4471 *He* whilst λ4686 *He*$^+$ remains strong can only be interpreted in one way, viz. as an increase in the degree of ionization. Again, during 1918 June there were broad bands due to *Fe*$^+$ and *Ti*$^+$ present; these disappeared towards the end of the month, and were not replaced by arc lines of these elements. Thus the second-stage ionization must have been completed before July, and this again involves a rise in temperature.

Later the temperature must have fallen again, for in 1926 we find the Balmer lines restored at the expense of the nebulium radiation. If we accept Eddington's view that metastable transitions are favoured by a weak field of radiation, then the only cause which can restore *H* at the expense of λ5007 is a decrease in the degree of ionization (for *H*, the ionization potential is 13·5 volts, and for O^{++} about 55). The

* Lunt, *Monthly Notices*, **79**, 416, 1919; **80**, 519 and 696, 1920; Hubble and Duncan, *loc. cit.*

† Eddington, *loc. cit.*

temperature cannot fall lower than the value prevailing in 1918, however, for λ 4471 does not reappear.

In general we may say, therefore, that

$$T_{1919} > T_{1926} > T_{1918}.$$

Now by 1926 the star had regained its original magnitude, and so, presumably, its original radius and temperature. If we assume that this was the case, we can find at once the density of the shell of gas; for the helium is evidently at least 90 per cent. ionized, and so we get for the fictitious electron pressure (using the known I.P. of helium),

$$p_e = 1{\cdot}7 \times 10^{-10} \text{ atm. for } T = 10{,}000^\circ \text{ A.}$$
$$p_e = 2{\cdot}8 \times 10^{-9} \text{ atm. for } T = 11{,}000^\circ \text{ A.}$$

Since $R = R_0$, this is equal to the original electron pressure in the shell, and corresponds to a region high up in the chromosphere of the star.

On the other hand, it has been suggested by Elvey * that the shell originates in the reversing layer of the star, *i.e.* where p_e is of the order 10^{-4} atm. This leads to $T_{1926} = 17{,}000^\circ$ A., which is a long way removed from the original temperature of the star. We have seen, however, that $T_{1926} > T_{1918}$, and so we are left with the following alternatives: either the star returns to its previous magnitude at a higher temperature than it had before, so that its radius is less than before the outburst; or the temperature never rises at any stage by more than 1000 degrees or so above normal, so that the whole phenomenon may be described as a sort of "cold explosion." Until this question is settled it is impossible to hope for a satisfactory theory of novæ; but it may be pointed out that Milne's † theory of the expulsion of atoms under radiation-pressure definitely requires only a small rise in temperature; a rise of more than a few thousand degrees would produce far higher velocities than any actually observed.

In conclusion, one further small point may be mentioned. Taking the values $p_e = 1{\cdot}7 \times 10^{-8}$, $T = 10{,}000^\circ$ A. to represent the state of affairs in 1926, we find $1 - x = 4 \times 10^{-7}$ for the fractional number of neutral H atoms present; and $x = 4 \times 10^{-5}$ for the fractional number of O^{++} atoms present.

If we choose instead $p_e = 10^{-4}$, $T = 17{,}000^\circ$ A., we get $1 - x = 10^{-4}$ for H, $x = 5 \times 10^{-3}$ for O^{++}.

In either case there is a very small relative concentration of the neutral hydrogen atoms; and this fact, taken in conjunction with the strength of the Balmer lines, shows that the hydrogen must be present in the cloud in great abundance compared with other elements. This explains why hydrogen does not disappear at the same time as Fe^+ and Ti^+, which have about the same ionization potential.

* *Nature*, **121**, 12, 1928.
† *Monthly Notices*, **86**, 459, 1926.

Physics Department, University of Leeds:
1928 *February.*

Photoelectric Observations of two Low-temperature B-type Stars.

By W. M. Smart, M.A., D.Sc.

(*Communicated by the Director, Cambridge Observatory.*)

§ 1. *Summary.*—The stars concerned are ζ Persei (visual magnitude 2·91, spectral type B1) and 9 Cephei (visual magnitude 4·87, spectral type B2).

The differences of magnitude between ζ Persei and β Arietis (2·72, A5), and between 9 Cephei and o Cassiopeiæ (4·74, B2), were measured by the photoelectric photometer of Cambridge Observatory.

The results are : ζ Persei is fainter by $0^m{\cdot}21$ photoelectrically than β Arietis, and 9 Cephei is fainter by $0^m{\cdot}63$ than o Cassiopeiæ.

From observations of several stars of different spectral type, a scale of photoelectric colour-index has been derived. It is thus possible by using this scale to determine the spectral type of any star whose visual magnitude is known.

From the observations of the two stars concerned it is concluded that ζ Persei behaves photoelectrically as a normal star of type A5, and 9 Cephei as a normal star of type Fo.

§ 2. During the past year regular observations of an experimental nature have been made by Mr. Green and the writer with the photoelectric photometer installed on the Sheepshanks coudé equatorial. This type of telescope possesses definite advantages for photoelectric work, as the cell (by a simple mechanical arrangement) and the electrometer are fixed in position. Of six cells experimented with, only one (a sodium cell filled with argon) was found to be comparatively free from the multitudinous defects and idiosyncrasies of this kind of instrument. The cells and the necessary fittings were supplied by Günther and Tegetmeyer of Brunswick, and the instrument was mounted on the polar axis tube by the Cambridge Scientific Instrument Company, who also supplied one of their standard string-electrometers. As the equipment is not yet complete, and as a new method of conducting the observations is still under consideration, this is not the occasion to give any detailed information with regard to the apparatus.

The applied voltage on the cell was generally about 280 volts. At 286 volts the cell showed signs of instability ; according to the makers the glowing potential is 300 volts. With ± 36 volts on the electrometer plates, the tension of the electrometer fibre was adjusted to give about 40 divisions (on the electrometer scale) to the volt. For most of the period during which the cell was in use, the leak and " dark current " were generally so small that for the purposes of this note they can be regarded as negligible. The cell, in fact, gave promise of reliability, which was very gratifying after months of patient experiments and mostly depressing experiences. Although no attempt was made to push the sensitivity of the apparatus to its utmost limit, a star of visual magnitude 5·0 and spectral type Ao gave a sweep of the electrometer

fibre of 10 scale divisions in about 40 seconds. The faintest star observed was 6 Sextantis, magnitude 6·0, spectral type A3.

§ 3. The experimental observations have been principally of two kinds. In the first, a pair of stars differing by a few tenths of a magnitude and close together in the sky have been observed on as many nights as were available, the object being to test the consistency (or otherwise) of the cell's performances. In the second, several stars of different spectral type were observed to see if a consistent scale of colour-index (photoelectric) could be derived. This latter work is the basis of the present investigation.

§ 4. In a recent paper * Greaves, Davidson, and Martin have drawn attention to the low colour-temperatures of several B-type stars. It occurred to the writer that photoelectric observations of these stars might be of value in assigning the equivalent spectral classes. Between 1928 December 5 and 14 two stars of the number were observed on 5 nights. Then supervened an interval of 3 or 4 weeks, during which abnormally bad weather conditions made it impossible to carry on the observations. It was then found that the cell had developed a leak, which was serious enough to bring the observations to a summary termination. It was intended to make a careful determination of the colour-indices of several stars of different spectral types, but under the circumstances this part of the programme had to be abandoned. However, the data obtained in previous experiments with the cell have been used. These results are as follows (the photoelectric and visual magnitudes of an Ao star are taken to be the same as in the photographic-visual system):—

Type	B2	Ao	Fo	Go	Ko
Colour-index (photoelectric)	−0·2	0·0	0·3	0·5	1·0

The two stars observed are ζ Persei (visual magnitude 2·91, spectral type B1) and 9 Cephei (4·87 and B2). β Arietis (2·72 and A5) was used as the comparison star for ζ Persei and o Cassiopeiæ (4·74 and B2) for 9 Cephei.

§ 5. The method of observing the magnitude difference between two stars X_1 and X_2 is as follows. The telescope is pointed to X_1, and the light from this star is allowed to enter the photoelectric cell. The rate of increase in potential of the collecting ring (connected to the electrometer fibre) of the cell due to emission of electrons from the photoelectric metal is found from the time T_1 required for the sweep of the fibre across, say, 10 divisions of the scale. Several observations are made in rapid succession with the aid of a stop-watch reading to $0^s{\cdot}1$. The telescope is then pointed to X_2, and the corresponding times T_2 are observed in a similar way. The first star is again observed, and the average time T_1 for both sets is obtained. The effects of progressive changes (if any) in the tension of the fibre and in the electrometer zero are thus eliminated.

The ratio of the incident light from X_1 and X_2 is equal to the ratio

* *M.N.*, **89**, 125, 1928.

of T_2 to T_1. We neglect any leak or drift due to the cell. If $m_1 - m_2$ is the difference of the photoelectric magnitudes of X_1 and X_2, then

$$m_1 - m_2 = 2{\cdot}5 \log T_1/T_2.$$

The following is a sample of the observations from the observing book:—

	β Arietis.	ζ Persei.	β Arietis.
	s	s	s
	14·8	17·6	14·0
	14·5	17·0	14·1
	14·0	17·0	14·0
	14·0	16·7	14·0
	14·3	16·7	13·8
	14·3	16·7	13·8
Mean	14·32	16·95	13·95

If m_1, m_2 denote the photoelectric magnitudes of ζ Persei and β Arietis respectively,

$$m_1 - m_2 = 2{\cdot}5 \, [\log 16{\cdot}95 - \log 14{\cdot}14],$$

from which

$$m_1 - m_2 = 0{\cdot}20.$$

The results for the 5 observing nights are summarised below. The magnitude of the first star of the pair is, in each instance, denoted by m_1 and of the second by m_2.

ζ Persei—β Arietis.

	Dec. 5.	Dec. 7.	Dec. 8.	Dec. 9.	Dec. 14.
No. of sets of observations	2	3	1	1	2
Mean value of $(m_1 - m_2)$	0·23	0·19	0·16	0·21	0·25

from which, for all the observations,

$$\mathbf{m_1 - m_2 = 0{\cdot}21.}$$

9 Cephei—o Cassiopeiæ.

	Dec. 7.	Dec. 8.	Dec. 9.	Dec. 14.
No. of sets of observations	1	3	2	2
Mean value of $(m_1 - m_2)$	0·73	0·65	0·55	0·64

from which, for all the observations,

$$\mathbf{m_1 - m_2 = 0{\cdot}63.}$$

The observations were made as far as possible with each pair of stars at not greatly differing altitudes. No correction has therefore been applied for atmospheric absorption depending on differences in altitude, nor for differences in spectral type. We have not yet been able to investigate this problem, but it is hardly likely that such a correction will affect appreciably our present results.

The difference between the visual magnitudes of ζ Persei and β Arietis is 0·19; the photoelectric difference is 0·21. We take these differences simply to be 0·2. If ζ Persei were a normal B-type star, it is seen from the colour-indices in §4 that this star ought to be $0^m{\cdot}35$ brighter photoelectrically than an A5 star of the same visual brightness, or $0^m{\cdot}15$ brighter than β Arietis. The observations show that it is $0^m{\cdot}2$ fainter photoelectrically than β Arietis. Its equivalent spectral type derived photoelectrically is thus about A5.

9 Cephei is fainter visually by $0^m{\cdot}17$ than o Cassiopeiæ, and fainter photoelectrically by 0·63. As the stars are spectroscopically of the same spectral type (B2), the difference (0·63 − 0·17), or 0·46, is due to the difference of photoelectric colour-index between a normal B2 star and a normal star of approximate spectral type Fo.

We infer from our observations that, photoelectrically, ζ Persei behaves as a normal star of type A5, and 9 Cephei as a normal star of approximate spectral type Fo. As far as ζ Persei is concerned, our result is in agreement with the conclusion of Greaves, Davidson, and Martin.*

Observatory, Cambridge:
1929 March 5.

Provisional Elements of the Binary System β.G.C. 12304.

By C. Luplau Janssen, M.Sc.

By the reduction of a series of micrometer measures of double stars made here at the Urania Observatory, it was remarked that the distance of the components of the double star β.G.C. 12304 (= Σ3001) had begun to decrease again after a constant increase from the first observation in 1832 until 1910; the later observations make the distances smaller. The change of the direction of motion was so sudden, that it obviously must be a very elongated orbit in which the companion is moving. Therefore a diagram was made containing the rather few measures which have been made of this pair, which is by no means difficult to measure. Many of the measures are discordant, but there cannot be the least doubt of the character of the motion. The diagram showed that it was possible to determine the centre of the orbit fairly well in spite of the shortness of the arc in consideration. Further, it came out that the apastron was situated in the part of the orbit covered by the observations. The elements were then easily derived by graphical methods, and if they have anything to do with the reality, the system is of the interesting γ Virginis type. Generally it has no meaning to publish elements based on so short an arc, but in this case we may be allowed to make an exception, as it seems to be an interesting system, and especially as a good number of observations during the years to come are necessary in order to get exact elements. The motion will

* *Loc. cit.*

now be quicker as the companion is expected to be in periastron towards 1976 and the eccentricity is 0·95.

The elements found are :

$$\begin{aligned} T &= 1976{\cdot}6 \\ P &= 177{\cdot}2 \quad \text{motion: direct.} \\ a &= 2''{\cdot}81 \\ e &= 0{\cdot}95 \\ i &= 58^\circ \\ \omega &= 94^\circ \\ \Omega &= 97^\circ \end{aligned}$$

All the observations which have been found in various publications have been collected in the following list. Possibly some single observations may have escaped me.

List of Observations.

1832·84	175°·0	2″·35	Σ.	1882·41	194°·1	2″·74	Hall.
1834·02	175·1	2·30	Dawes.	1891·03	195·6	2·60	HΣ.
1834·95	173·8	2·50	Smyth.	1892·88	195·1	2·81	Jones.
1844·30	179·8	2·43	Maedler.	1895·84	193·9	2·76	Collins.
1851·87	183·4	2·70	OΣ.	1897·67	198·3	3·13	Seabroke.
1852·65	185·0	2·20	Maedler.	1902·02	197·8	2·61	Maw.
1855·95	184·8	2·54	Δ.	1902·20	201·8	3·00	Doberck.
1856·92	186·8	2·53	Jacob.	1904·64	197·3	2·97	Thiele (phot.).
1858·44	187·2	2·47	Secchi.	1908·69	200·0	2·93	Doberck.
1858·62	186·9	2·60	Morton.	1910·47	198·5	3·11	Fox.
1861·87	187·3	2·62	Maedler.	1910·83	201·4	2·75	Voûte.
1864·68	187·1	2·47	Δ.	1911·89	200·0	3·01	Storey.
1874·91	189·9	2·65	Gledhill.	1912·80	203·5	2·98	Jameson.
1875·18	192·1	2·45	Δ.	1912·87	202·0	3·12	Storey.
1877·75	191·7	2·48	Doberck.	1921·73	203·6	2·92	Doberck.
1882·04	192·6	2·59	Jed.	1928·69	210·2	2·78	Luplau Janssen.

The elements represent the observations fairly well.

The hypothetical parallax of the system is

$$p_h = 0''{\cdot}089$$

Urania Observatory,
Copenhagen.

Variability of the Earth's Rotation. By Sir F. W. Dyson and R. T. Cullen. (Plates 7, 8.)

In the *Monthly Notices*, **89**, 260, attention was drawn incidentally to an approximate correspondence between the minor fluctuations of the Moon's and Sun's longitudes. The agreement seemed so much

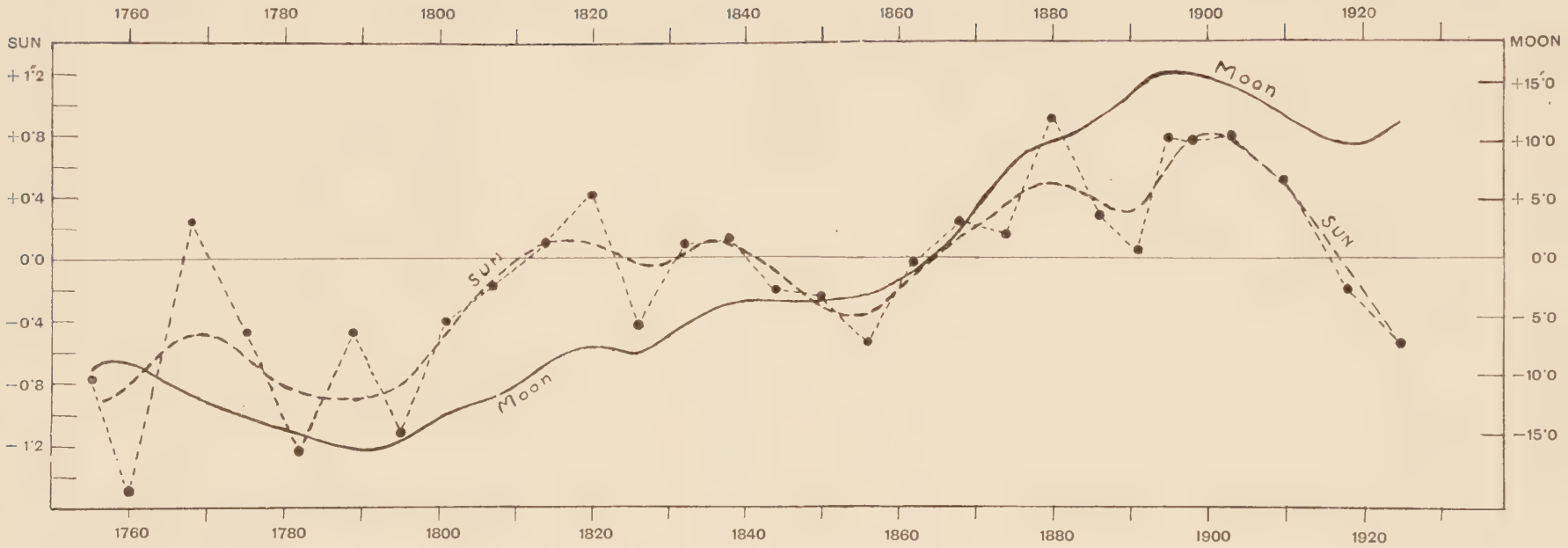
more satisfactory when the Sun's longitude was determined entirely from declinations, that it seemed worth while to give the comparison in more detail.

The substance of this paper is shown in two diagrams. In Plate 7 are compared the differences of the Sun's longitude from Leverrier's tables (Newcomb, *Astronomical Constants*, p. 30) and later observations, with the differences of the Moon's longitude from Brown's theory without the empirical term. This should be compared with Brown's diagram in vol. iii. of the *Publications of the Yale University Observatory*. There is a correspondence in the minor fluctuations, and there would be a very close accordance if the mean motion of Sun or Moon could be altered for the last 170 years. But bearing in mind the results obtained from occultations in the seventeenth century, it does not seem possible to alter the mean motion of either Sun or Moon sufficiently.

The minor fluctuations of the Moon's longitude are compared with those of the Sun in Plate 8. The tabular places of the Moon as given by Brown are corrected slightly (*Monthly Notices*, **83**, 367), and those of the Sun by applying $-0''{\cdot}17 - {\cdot}56T + 0''{\cdot}78T^2$ to Leverrier's tables. For the Sun the smoothed curve was drawn through the values obtained by taking mean and mean again of consecutive observations. Readings taken at intervals of five years from the smoothed curves drawn through the residuals in longitude for the Sun and the Moon are as follows:—

Longitude.

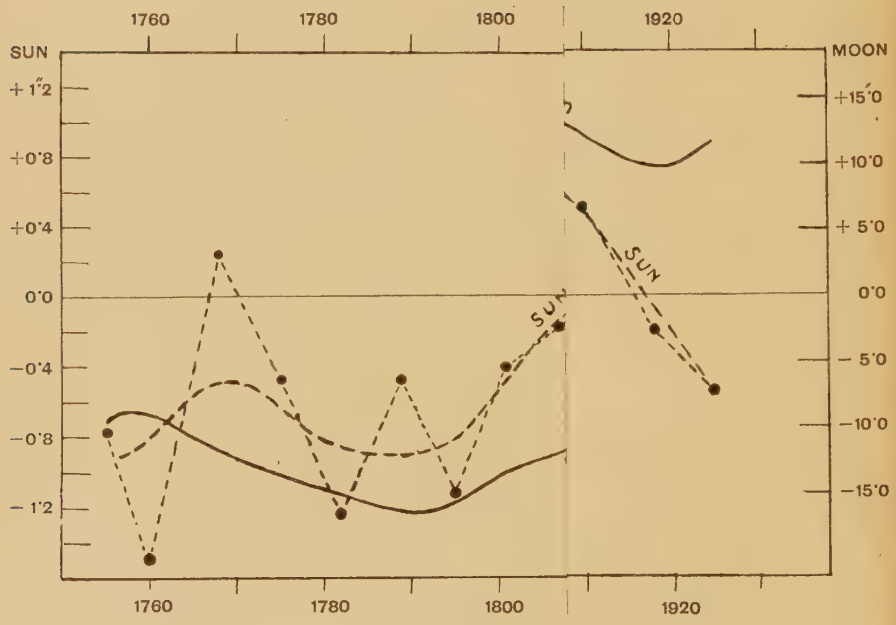
Year.	Tabular—Observed.		Year.	Tabular—Observed.	
	Sun.	Moon.		Sun.	Moon.
	″	″		″	″
1755	+·01	−0·5	1840	−·11	+0·9
60	+·06	+1·3	45	−·32	−0·7
65	+·35	+0·6	50	−·49	−2·1
70	+·42	−0·3	55	−·56	−3·5
75	+·22	−0·7	60	−·45	−3·9
80	−·17	−1·3	65	−·23	−3·4
85	−·34	−1·8	70	−·04	−1·3
90	−·38	−2·7	75	+·14	+1·7
95	−·40	−2·0	80	+·27	+2·2
1800	−·26	−0·2	85	+·14	+2·7
05	−·03	+0·4	90	+·02	+3·7
10	+·17	+1·1	95	+·32	+4·5
15	+·28	+2·5	1900	+·52	+3·3
20	+·20	+2·5	05	+·42	+1·2
25	−·05	+0·6	10	+·17	−1·1
30	−·09	+0·9	15	−·13	−3·3
35	−·07	+1·6	20	−·40	−3·8
			1925	−·60	−1·8



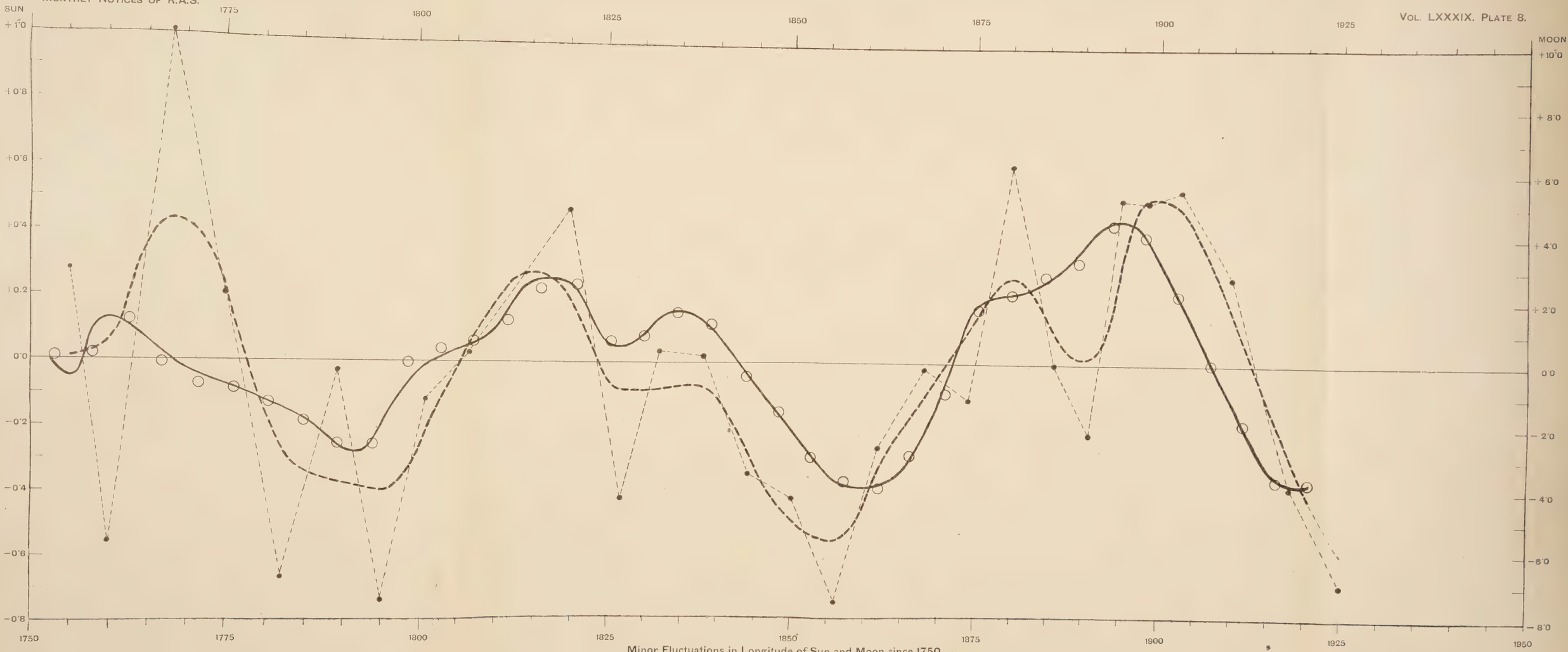
Longitude of Sun and Moon since 1750.

"Leverrier – observed" for Sun. "Brown (without G.E.T.) – observed" for Moon.

Ratio 13·3 to 1.


1760
1780
1800
1920
SUN
+1″2
+0·8
+0·4
0·0
−0·4
−0·8
−1·2
MOON
+15′0
+10′0
+5′0
0′0
−5′0
−10′0
−15′0
SUN
SUN

"Leverrier



Minor Fluctuations in Longitude of Sun and Moon since 1750.
Sun = broken curve and dots.
Moon = smooth curve and circles.
Ratio 10 to 1.

Comparison of these values gives

$$10{\cdot}1\Delta\odot = \Delta\mathbb{D} - 0''{\cdot}03,$$

or the minor fluctuations of the Moon are ten times those of the Sun, agreeing with Dr. Fotheringham (*Monthly Notices*, **87**, 166, 1926).

The agreement of the two curves is striking—the main differences being that the Sun's longitude is 0″·25 too small from 1825 to 1855, and too large by 0″·5 from 1901 to 1916. These are very small quantities as they depend on accordance in the observations of the Sun's declination in Spring and Autumn to 0″·1 and 0″·2 respectively.

The observational grounds on which declinations are to be preferred to right ascensions in determining the Sun's longitude in the past are the existence of large personalities in observing transits of the Sun. Since the introduction of the travelling micrometer, this uncertainty has been largely eliminated.

Preliminary Values of the Variation of Latitude at Greenwich in 1928.

(*Communicated by the Astronomer Royal.*)

This paper contains the preliminary values of the latitude variation derived from observations made during 1928 with the Cookson Floating Telescope.

The star places used were those derived from the final discussion of the observations for the years 1919-26. Corrections for proper motion and temperature have been applied to each pair of stars. Owing to cloudy nights only a comparatively small number of plates were taken in June, November, and December.

The variation of latitude has again been very small, the extreme range from 1925·0 to 1929·0 being only 0″·15.

The following table gives the values of the latitude variation, in the sense mean latitude *minus* observed latitude :—

Date.	Variation.	Number of Pairs.
	″	
1928·033	+·026	109
·171	+·045	109
·261	+·008	71
·361	+·093	88
·437	−·026	29
·505	−·022	70
·574	−·074	95
·668	+·008	144
·776	+·021	137
·900	+·041	51
1929·060	−·015	90

From a smooth curve drawn to represent the observed results as closely as possible, the following corrections, applicable to observed north polar distances, were derived:—

1928·0	+″·04	1928·6	−″·04
28·1	+·06	28·7	−·01
28·2	+·06	28·8	+·03
28·3	+·04	28·9	+·04
28·4	+·02	1929·0	+·03
28·5	−·03		

The final reduction of the observations from 1919 to 1927 was published in November last.

Royal Observatory, Greenwich:
1929 *February* 27.

Occultations of Stars by the Moon, observed at Tomsk University Observatory during the Year 1928. By Professor N. Goryatscheff.

(*Communicated by the Secretaries.*)

(ϕ = N. 56° 28′ 07″; l = 5^h 39^m $47^s{\cdot}4$ East of Greenwich.)

Date (U.T.).	Local Sidereal Time.	Star.	Mag.	Phase.
1928.	h m s			
III. 27·611144	8 38 56·8	B.D.+24°·931	8·2	D
III. 27·689086	10 31 29·4	B.D.+23°·1043	8·5	D
III. 27·723140	11 20 39·6	B.D.+24°·973	7·7	D
III. 29·592155	8 19 24·7	B.D.+24°·1755	6·8	D
VIII. 3·784189	21 17 24·6	69 Aquarii	5·6	R
IX. 6·802821	23 58 21·6	B.D.+22°·754	8·7	R
IX. 6·864536	1 27 28·4	B.D.+22°·762	8·7	R
IX. 7·788753	23 41 59·3	399 B Tauri	7·2	R
IX. 7 790294	23 44 12·9	B.D.+24°·965	7·9	R
IX. 7·809056	0 11 18·3	132 Tauri	5·6	R
IX. 8·803947	0 07 52·3	B.D.+25°·1459	8·7	R
IX. 8·841162	1 01 36·5	B.D.+25°·1482	7·5	R
IX. 8·877566	1 54 10·4	37 Geminorum	5·7	R
X. 3·654634	22 10 50·2	227 B Tauri	5·9	R

Apparent Places of Occulted Stars of Magnitude $> 6{\cdot}5$.

Star.	α.	δ.	Authority.
	h m s	° ′ ″	
B.D.+24°·931	5 39 01·13	+24 03 15·4	Ox. ph. 25°·16037.
B D.+23°·1043	5 43 02·83	+23 51 35·3	Par. ph. 23° cl. 567·236; 24° cl. 147·368.
B.D.+24°·973	5 44 55·33	+24 12 04·4	Ox. ph. 25°·16148.
B.D.+24°·1755	7 39 37·70	+24 25 03·4	Ox. ph. 25°·30136.
B.D.+22°·754	4 43 10·37	+22 10 38·7	Par. ph. 22° cl. 1296 rep.; 23° cl. 371 rep.
B.D.+22°·762	4 46 24·99	+22 34 29·1	Par. ph. 22° cl. 396 rep.; 23° cl. 371 rep.
399 B Tauri	5 43 31·87	+24 39 46·7	W.Z.C. Hedrick.
B.D.+24°·965	5 43 33·55	+24 38 16·5	Ox. ph.+25°·16397.
B.D.+25°·1459	6 46 20·96	+25 21 04·5	Ox. ph. 25°·25147; 26°·19271.
B.D.+25°·1482	6 48 41·37	+25 24 05·7	Ox. ph. 25°·25176; 26°·19298.

Observers.—Professor N. Goryatscheff (all), and B.D. + 24°·973 (III. 27) Professor N. Goryatscheff and Stud. Neshivoy.

Instrument.—Zeiss telescope, 90 mm. aperture; magnifying power 90. Observations made with eye and ear.

Tomsk, Siberia, U.S.S.R.,
University Astronomical Observatory:
1929 *January* 21.